RATIONAL DILATION

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0.1. **Introduction.** Hi, I'm here to talk to you today, about the rational dilation problem.

1. The Rational Dilation Problem

1.1. **Basics.** Firstly, what is a dilation? If we've got some Hilbert space operator $T \in B(K)$, a dilation of T is an operator $S \in B(H)$ (where H is a Hilbert space containing K), such that

$$P_K S|_K = T\,.$$

Here, P_K denotes the projection onto K, and $|_K$ denotes the restriction to K. Equivalently, a dilation is something that looks like

$$S = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{T} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

If we've got some compact $X \subseteq \mathbb{C}$, we say that an operator $T \in B(K)$ has a *rational ∂X-dilation*, if there exists a bigger Hilbert space $H \supseteq K$, and a normal operator $S \in B(H)$, such that $\sigma(S) \subseteq \partial X$, and for all $r \in \mathcal{R}(X)$,

$$P_K\, r(S)|_K = r(T)$$

So, what does that mean? For S to be a *rational* ∂X-dilation, rather than just a dilation, we also need that S is normal, $\sigma(S) \subseteq \partial X$, and that

$$r(S) = r\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{T} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{r(T)} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

for all $r \in \mathcal{R}(X)$.

The idea behind a dilation is to take a badly behaved operator, and find a larger, better behaved operator, that has our badly behaved operator as a block on its diagonal (normal operators are particularly well behaved). So, when we go further, and look for a rational ∂X-dilation, we're looking for a dilation that dilates not just our operator, but rational functions of our operator too. I should also say that, although the condition about the spectrum of S looks kinda arbitrary, it actually falls out of the theory quite naturally.

Now, von Neumann showed that if T has a rational ∂X-dilation, then

(1) $\sigma(T) \subseteq X$, and
(2) for all $r \in \mathcal{R}(X)$, $\|r(T)\|_{B(K)} \leq \|r\|_{C(X)}$

The latter is the famous *von Neumann inequality*. We say that X is a *spectral set* for T when both these conditions hold, so "X is a spectral set for T" is a necessary condition for "T has a rational ∂X-dilation".

So, a natural question to ask is, "Is it condition sufficient?", which leads to the rational dilation conjecture:

Conjecture. *An operator $T \in B(H)$ has a rational ∂X-dilation, if and only if X is a spectral set for T.*

1.2. **Sz.-Nagy's Dilation Theorem.** A classical result in this field is Sz.-Nagy's dilation theorem ([Pau02, Thm. 4.3]). Sz.-Nagy looked at the special case where X is the unit disc, $\mathbb{D}$, with boundary $\mathbb{T}$, the unit circle.

In this case, our definitions have some nice equivalent statements. Saying $\mathbb{D}$ is a spectral set for T, is the same as saying that $\|T\| \leq 1$, that is, T is a contraction. Also, S is normal and has spectrum on $\mathbb{T}$, if and only if S is unitary. So, it's enough to show that all contractions have a rational unitary dilation.

To do this, first, we show that every contraction has an isometric dilation, so every contraction dilates to an isometry. We know that if $D_T = (1-T^*T)^{1/2}$, then

$$\begin{aligned}
\|Tx\|^2 + \|D_T x\|^2 &= \langle Tx, Tx\rangle + \langle D_T x, D_T x\rangle \\
&= \langle T^*Tx, x\rangle + \left\langle D_T^2 x, x\right\rangle \\
&= \left\langle \left(T^*T + D_T^2\right) x, x\right\rangle \\
&= \langle Ix, x\rangle = \|x\|^2
\end{aligned}$$

so if we define

$$V := \begin{pmatrix} \boxed{T} & 0 & 0 & 0 & \cdots \\ D_T & 0 & 0 & 0 & \ddots \\ 0 & I & 0 & 0 & \ddots \\ 0 & 0 & I & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

then

$$\|Vx\|^2 = \underbrace{\|Tx_1\|^2 + \|D_T x_1\|^2}_{\|x_1\|^2} + \|x_2\|^2 + \|x_3\|^2 + \cdots = \|x\|^2$$

so we can see that this is an isometry. Also, because the operator is lower triangular, we can see that

$$r(V) = \begin{pmatrix} \boxed{r(T)} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

The next thing to do is show that every isometry has a unitary dilation. If we define

$$U = \begin{pmatrix} \boxed{V} & I - VV^* \\ 0 & V^* \end{pmatrix}$$

then we can show that U is unitary, and because it's upper triangular,

$$r(U) = \begin{pmatrix} \boxed{r(V)} & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \boxed{r(T)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

so U is a normal $\mathbb{T}$-dilation of T, so T has a normal $\mathbb{T}$-dilation.

A generalisation of this result, due to Berger, Foias and Lebow ([Pau02, Thm. 4.4]), shows that the rational dilation conjecture also holds whenever our region $X \subseteq \mathbb{C}$ is simply connected – that is, whenever X has no holes.

1.3. **Completely bounded maps.** Most modern approaches to rational dilation are based on completely bounded maps – Arveson showed that the rational dilation conjecture was equivalent to a result about completely bounded maps.

First, note that we can define a homomorphism $\pi : \mathcal{R}(X) \to B(K)$ by

$$\pi(r) = r(T)$$

We can then see that for X to be a spectral set for T, we need

$$\|\pi(r)\|_{B(K)} = \|r(T)\|_{B(K)} \leq \|r\|_{C(X)}$$

but this is the same as saying $\|\pi\| \leq 1$, thinking about π as a linear map, and using the standard norm for linear maps. So, we know that if X is a spectral set for T, this is the same as saying π is contractive.

Now, we'd like to find some way of describing rational ∂X-dilations in terms of π too, and to do this, we need to define *completely contractive* maps.

Suppose we have two C^*-algebras, C and C', a vector subspace $M \subseteq C'$, and a bounded linear map $\phi : M \to C$. We can then define C^*-algebras M_nC and M_nC', which consist of $n \times n$ matrices, whose elements come from C and C', so

$$M_nC := \left\{ \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} \middle| \, c_{ij} \in C \text{ for all } i\, j \right\}$$

We can then define a linear map $\phi_n : M_nM \to M_nC$ by

$$\phi_n \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \dots & m_{nn} \end{pmatrix} = \begin{pmatrix} \phi(m_{11}) & \dots & \phi(m_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(m_{n1}) & \dots & \phi(m_{nn}) \end{pmatrix}$$

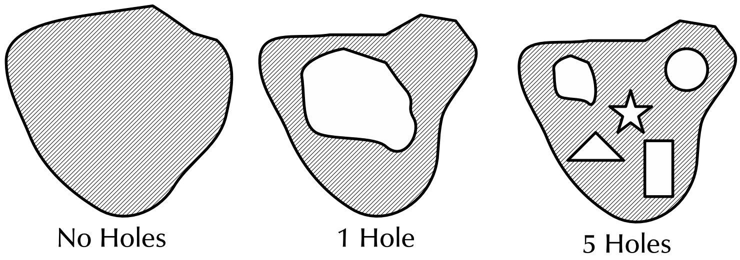

FIGURE 1.1. Planar domains with various numbers of holes

We know that for each $n \in \mathbb{N}$, ϕ_n is a bounded linear operator, and so we define the *complete norm* of ϕ by

$$\left\|\phi\right\|_{cb} := \sup_{n\in\mathbb{N}} \left\|\phi_n\right\|$$

We say that ϕ is *completely bounded* if $\left\|\phi\right\|_{cb} < \infty$, and ϕ is *completely contractive* if $\left\|\phi\right\|_{cb} \leq 1$.

Another useful idea is *complete positivity*. We say a map is *completely positive* if ϕ_n is positive for all n.

Arveson proved that rational dilation was equivalent to a number of other conditions:

Theorem. *The following are equivalent:*

(1) *T has a rational ∂X-dilation*
(2) *The homomorphism $\pi : \mathcal{R}(X) \to B(K)$ is completely contractive*
(3) *The homomorphism $\widetilde{\pi} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to B(K)$ is competely positive.*

The rational dilation question then becomes: if π is contractive, does that mean it's completely contractive?

1.4. **Multiply connected domains.** We're interested in spaces that are *not* simply connected. We say a compact set $X \subseteq \mathbb{C}$ is an n-holed domain if its boundary, ∂X, has $n + 1$ components; this is kinda unintuitive, but if we look at Figure 1.1, we can see that this is really just the same as saying X has n holes in it.

Agler showed that if X is a 1-holed domain with smooth boundary, then the rational dilation conjecture holds ([Agl85]). However, later work showed that the rational dilation conjecture doesn't hold on 2-holed domains: Agler, Harland and Raphael ([AHR08]) found an example of a 2-holed domain X and an operator T (in fact, their operator was just a 4×4 matrix), that fails the conjecture on X. Dritschel and McCullough then went on to show that if we have any 2-holed domain X, with smooth boundary, then there exists an operator T, that fails the conjecture on X ([DM05]).

It's suspected that that the rational dilation conjecture doesn't hold on any 2-or-more-holed domain, but Dritschel and McCullough's proof doesn't easily generalise to all n-holed domains. However, I've been able to extend their proof to 2-or-more-holed domains with certain useful symmetries ([Pic08]). I'll be talking some more about these symmetries a little later, but for now, all I'm going to say is that all 2-holed domains have these symmetries, so the two results are related.

1.5. **Operators.** Something I should quickly mention, is that not all of the research in this area has focused on the space X. There's also been some research into what kinds of operators will work too. For starters Misra, Paulsen and Agler, between them, showed that if our operator, T, is a 2×2 matrix, then rational dilation holds – so no matter how badly behaved X is, if T is a 2×2 matrix, and X is a spectral set for T, then T has a rational ∂X-dilation. On the other hand, as I mentioned before, Agler, Harland and Raphael's couterexample is a 4×4 matrix, so clearly rational dilation can't hold on all 4×4 matrices. At the moment, we don't know about 3×3 matrices, although in [AHR08], the authors conjecture that rational dilation *does* hold for all 3×3 matrices, based on numerical evidence, although – so far as I know – there's no rigourous proof. You can find out more about this approach in [AHR08].

The holy grail in this field, is to find a simple description of which operators have a rational ∂X-dilation, but I think we're quite a long way from being able to do this.

2. The counterexample

Now, although this talk is about the result in [Pic08], I'm taking a different approach to the one I take there. The approach I'm taking here is based on generalised Herglotz representations. It should be more illustrative, and is closely related to the approach I take in the paper, but I can't promise it's right.

For the construction I use, X needs to be conformally equivalent to a domain which is symmetric about the real axis. Having said that though, it'll make the calculations look simpler to just assume X *is* symmetric about the real axis – rather than just conformally equivalent to something that is – so I'm going to make this assumption. See figure 2.1 on the next page for an example of such a domain.

2.1. **Agler-Herglotz Representations.** I said I'd be talking about generalised Herglotz representations. The classical Herglotz representation theorem is a theorem about functions on the disc (although it's sometimes stated for half planes). It goes as follows:

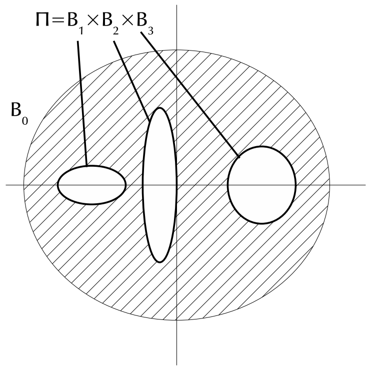

FIGURE 2.1. An example of a symmetric domain

Theorem. *g is a positive harmonic function on $\mathbb{D}$, if and only if there is a positive measure μ on $\mathbb{T}$ such that*

$$g(z) = \int_{\mathbb{T}} \mathbb{P}_w(z) d\mu(w)\,,$$

where $\mathbb{P}$ is the Poisson kernel.

We can write this in an equivalent way for holomorphic functions:

Corollary. *f is a holomorphic function on $\mathbb{D}$, with positive real part, and $f(0) > 0$, if and only if there is a positive measure μ on $\mathbb{T}$ such that*

$$f(z) = \int_{\mathbb{T}} h_w(z) d\mu(w)\,,$$

where h is the Herglotz kernel.

Here, the theorem essentially follow from Dirichlet's principle (the measure μ is the "boundary value" of f in some sense), and the corollary follow from the fact that *on simply connected spaces*, harmonic functions are the real parts of holomorphic functions.

This isn't quite what we want. We're working in a space X, so we want a Herglotz-like representation on X, rather than $\mathbb{D}$ (I call these representations *Agler-Herglotz representations*).

On X, the Dirichlet principle still holds, and we have a representation like the first one:

Theorem. *g is a positive harmonic function on X, if and only if there is a positive measure μ on ∂X such that*

$$g(z) = \int_{\partial X} \mathbb{P}_w(z) d\mu(w)\,,$$

where $\mathbb{P}$ is the Poisson kernel.

However, as it is, the corollary doesn't quite hold. The problem is, that not all positive harmonic functions correspond to holomorphic functions (think about something like $\log|z|$ on the annulus. It's a perfectly good harmonic function, but it's the real part of $\log(z)$, which is not a single-valued holomorphic function).

We can get round this, though. If our harmonic function g *is* the real part of a holomorphic function, then we're done. The question is, which measures μ correspond to holomorphic functions?

For reasons that are too complicated to discuss here, a necessary condition for μ to correspond to a holomorphic function, is that it must be non-zero on each component of ∂X (we have a better condition, that's necessary and sufficient, but it's complicated). So, measures have to touch each boundary component (we call them $B_0, \ldots, B_n$).

We also know that holomorphic measures form a cone, so we can add them together. This means, if we can find a set of simple "building block" measures, we can build all other measures out of them. The simplest measure that's supported on at least one point in each point of $B_0, \ldots, B_n$, is the measure that's supported at exactly one point in each of $B_0, \ldots, B_n$. So, for each point in $P = B_0 \times \cdots \times B_n$ (i.e, for each tuple of points from $B_0, \ldots, B_n$), we have a measure μ_p, and a corresponding holomorphic function h_p.

These simplest functions h_p are also "extremal", in the sense of Krein-Milman, or Choquet, so we can indeed build all holomorphic functions as a combination of them. This gives us an Agler-Herglotz representation:

Corollary. *f is a holomorphic function on X, with positive real part, and $f(0) > 0$, if and only if there is a positive measure μ on P such that*

$$f(z) = \int_P h_w(z) d\mu(w) .$$

The real reason this is interesting though, is h_p. We can break down h_p into $h_p^0 + h_p^1 + \cdots + h_p^n$, where μ_p^i is a point mass at p_i, so μ_p^1 is a point mass at p_1, μ_p^2 is a point mass at p_2, etc. I'll be using these h_p^is to construct our counterexample.

2.2. **Matrix-Valued Agler-Herglotz representations.** It turns out that we can also define a Matrix-Valued analogue of the Agler-Herglotz representation, provided rational dilation holds on our set X.

Suppose we have a positive matrix-valued holomorphic function $F : X \to M_n$. So long as $(\Re F)(T) \geq 0$ for all X-spectral operators T, we have an Agler-Herglotz like representation:

$$F_{ij}(z) = \int_P h_w(z) d\mu_{ij}(w) .$$

for some positive, matrix-valued measure $\left(\mu_{ij}\right)$.

But now think of the contrapositive. If we have an F with no such representation, then there must be some X-spectral operator T, such that

$(\Re F)(T) \ngeq 0$. This is a problem, as $(\Re F)(T) = \widetilde{\pi_n}(\Re F)$, so this would mean that $\widetilde{\pi}$ was not completely positive.

This would give a counterexample to rational dilation

2.3. **The counterexample.** So, how do we construct our counterexample? First, let's look at a nice holomorphic function, if we define

$$H_p = \begin{pmatrix} h_p^0 & 0 \\ 0 & h_{\overline{p}}^0 \end{pmatrix} + \begin{pmatrix} h_p^1 & 0 \\ 0 & h_{\overline{p}}^1 \end{pmatrix} + \cdots + \begin{pmatrix} h_p^n & 0 \\ 0 & h_{\overline{p}}^n \end{pmatrix} = \begin{pmatrix} h_{\mu_p} & 0 \\ 0 & h_{\mu_{\overline{p}}} \end{pmatrix}$$

then this is a harmonic function, and it's the real part of a holomorphic function. It has a perfectly good matrix Agler-Herglotz representation, as we can take $\mu = \begin{pmatrix} \mu_p & 0 \\ 0 & \mu_{\overline{p}} \end{pmatrix}$.

So, how do we break it? We do something like this:

$$\widetilde{H_p} = \begin{pmatrix} h_p^0 & 0 \\ 0 & h_{\overline{p}}^0 \end{pmatrix} + U_1^* \begin{pmatrix} h_p^1 & 0 \\ 0 & h_{\overline{p}}^1 \end{pmatrix} U_1 + \cdots + U_n^* \begin{pmatrix} h_p^n & 0 \\ 0 & h_{\overline{p}}^n \end{pmatrix} U_n$$

for some unitary matrices $U_1, \ldots, U_n$. We can choose p, and our unitaries carefully, so that it's not possible to give this an Agler-Herglotz representation – its terms have been "twisted". Therefore, rational dilation fails on X, for any suitably symmetric X.

References

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