RATIONAL DILATION

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0.1. **Introduction.** Since this talk is being given for a mixed audience, it will be in two parts.

The first part's aimed at a general audience, so may seem a bit simplistic to the more experienced mathematicians here. I'll explain the background and motivation for the rational dilation problem, and give a summary of known results, including the statement of my own result. Please interrupt me during this part if there's anything you don't understand.

The second part's aimed at more experienced operator theorists. I'll be explaining a little more about how my result comes about, and the connections to the theory of Schur-Agler classes.

1. The Rational Dilation Problem

1.1. **Basics.** We'll startout with some definitions:

Throughout this talk, we'll be assuming X is a compact subset of $\mathbb{C}$, with ∂X as its boundary. We define

$$C(X) := \{f : X \to \mathbb{C} \,|\, f \text{ continuous on } X\}$$

and define $\mathcal{R}(X) \subseteq C(X)$ by

$$\mathcal{R}(X) = \left\{ \frac{p}{q} \,|\, p, q \text{ are polynomials, and } q \text{ has no zeros on } X \right\}$$

so $\mathcal{R}(X)$ is the set of all rational functions that are continuous on X, or equivalently, that have no poles on X, so all its poles are outside X. We know $C(X)$ is a C^*-algebra under the supremum norm, so $\mathcal{R}(X)$ is a normed algebra under this norm.

If $K \subseteq H$ are Hilbert spaces, we define

$$B(H) := \{T : H \to H \,|\, T \text{ is a bounded, linear operator}\}$$

and we define $P_K \in B(H)$ as the orthogonal projection operator onto K. If $T \in B(H)$, then $T|_K$ is the *restriction* of T to K, so it's just T, but though of as acting on K, rather than H. We define the *spectrum* of an operator T by

$$\sigma(T) := \{\lambda \in \mathbb{C} \,|\, T - \lambda I \text{ is not invertible}\}$$

We say an operator N is normal if it commutes with its adjoint, so $NN^* = N^*N$, we say V is an isometry if $\|Vx\| = \|x\|$, and we say U is unitary if $U^*U = I = UU^*$.

Hopefully, most of these definitions should be familiar.

Now, if we've got some compact $X \subseteq \mathbb{C}$, we say that an operator $T \in B(K)$ has a *rational ∂X-dilation*, if there exists a bigger Hilbert space $H \supseteq K$, and a normal operator $N \in B(H)$, such that $\sigma(N) \subseteq \partial X$, and for all $p/q \in \mathcal{R}(X)$,

$$P_K\, p(N)\, q(N)^{-1}|_K = p(T)\, q(T)^{-1}$$

So, what does that mean? If we've got some operator T, a dilation S of T is something that looks like

$$S = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{T} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

so we've got T as a block on the diagonal, and some other stuff elsewhere. Now, for N to be a *rational* ∂X-dilation, we also need that N is normal, $\sigma(N) \subseteq \partial X$, and that

$$r(N) = r\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{T} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \boxed{r(T)} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

for all $r \in \mathcal{R}(X)$. The idea behind a dilation is to take a badly behaved operator, and find a larger, better behaved operator, that has our badly behaved operator as a block on its diagonal (normal operators are particularly well behaved). So, when we go further, and look for a rational ∂X-dilation, we're looking for a dilation that dilates not just our operator, but rational functions of our operator too. I should also say that, although the condition about the spectrum of N looks kinda arbitrary, it actually falls out of the theory quite naturally.

Now, von Neumann showed that if T has a rational ∂X-dilation, then

(1) $\sigma(T) \subseteq X$, and
(2) for all $p/q \in \mathcal{R}(X)$, $\left\|p(T)\, q(T)^{-1}\right\|_{B(K)} \leq \left\|p/q\right\|_{C(X)}$

We say that X is a *spectral set* for T when both these conditions hold, so "X is a spectral set for T" is a necessary condition for "T has a rational ∂X-dilation".

So, a natural question to ask is, "Is this condition sufficient?", which leads to the rational dilation conjecture:

Conjecture. *An operator $T \in B(H)$ has a rational ∂X-dilation, if and only if X is a spectral set for T.*

1.2. **Sz.-Nagy's Dilation Theorem.** A classical result in this field is Sz.-Nagy's dilation theorem ([Pau02, Thm. 4.3]). Sz.-Nagy looked at the special case where X is the unit disc, $\mathbb{D}$, with boundary $\mathbb{T}$, the unit circle.

In this case, saying $\mathbb{D}$ is a spectral set for T, is the same as saying that $\|T\| \leq 1$, that is, T is a contraction. Also, N is normal and has spectrum on

$\mathbb{T}$, if and only if N is unitary. So, it's enough to show that all contractions have a rational unitary dilation.

To do this, first, we show that every contraction has an isometric dilation, so every contraction dilates to an isometry. We know that if $D_T = (1-T^*T)^{1/2}$, then

$$\begin{aligned} \|Tx\|^2 + \|D_T x\|^2 &= \langle Tx, Tx \rangle + \langle D_T x, D_T x \rangle \\ &= \langle T^*Tx, x \rangle + \left\langle D_T^2 x, x \right\rangle \\ &= \left\langle \left(T^*T + D_T^2 \right) x, x \right\rangle \\ &= \langle Ix, x \rangle = \|x\|^2 \end{aligned}$$

so if we define

$$V := \begin{pmatrix} \boxed{T} & 0 & 0 & 0 & \cdots \\ D_T & 0 & 0 & 0 & \ddots \\ 0 & I & 0 & 0 & \ddots \\ 0 & 0 & I & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

then

$$\|Vx\|^2 = \underbrace{\|Tx_1\|^2 + \|D_T x_1\|^2}_{\|x_1\|^2} + \|x_2\|^2 + \|x_3\|^2 + \cdots = \|x\|^2$$

so we can see that this is an isometry. Also, because the operator is lower triangular, we can see that

$$r(V) = \begin{pmatrix} \boxed{r(T)} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

The next thing to do is show that every isometry has a unitary dilation. If we define

$$U = \begin{pmatrix} \boxed{V} & I - VV^* \\ 0 & V^* \end{pmatrix}$$

then we can show that U is unitary, and because it's upper triangular,

$$r(U) = \begin{pmatrix} \boxed{r(V)} & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \boxed{r(T)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

so U is a normal $\mathbb{T}$-dilation of T, so T has a normal $\mathbb{T}$-dilation.

A generalisation of this result, due to Berger, Foias and Lebow ([Pau02, Thm. 4.4]), shows that the rational dilation conjecture also holds whenever our region $X \subseteq \mathbb{C}$ is simply connected – that is, whenever X has no holes.

1.3. **Completely bounded maps.** Arveson showed that the rational dilation conjecture was equivalent to a result about so-called completely bounded maps.

First, note that we can define a homomorphism $\pi : \mathcal{R}(X) \to B(K)$ by

$$\pi(p/q) = p(T)\,q(T)^{-1}$$

We can then see that for X to be a spectral set for T, we need

$$\left\|\pi(p/q)\right\|_{B(K)} = \left\|p(T)\,q(T)^{-1}\right\|_{B(K)} \leq \left\|p/q\right\|_{C(X)}$$

but this is the same as saying $\|\pi\| \leq 1$, thinking about π as a linear map, and using the standard norm for linear maps. So, we know that if X is a spectral set for T, this is the same as saying π is contractive.

Now, we'd like to find some way of describing rational ∂X-dilations in terms of π too, and to do this, we need to define *completely contractive* maps.

Suppose we have two C^*-algebras, C and C', a vector subspace $M \subseteq C'$, and a bounded linear map $\phi : M \to C$. We can then define C^*-algebras M_nC and M_nC', which consist of $n \times n$ matrices, whose elements come from C and C', so

$$M_nC := \left\{ \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} \,\middle|\, c_{ij} \in C \text{ for all } i\,j \right\}$$

We can then define a linear map $\phi_n : M_nM \to M_nC$ by

$$\phi_n \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \dots & m_{nn} \end{pmatrix} = \begin{pmatrix} \phi(m_{11}) & \dots & \phi(m_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(m_{n1}) & \dots & \phi(m_{nn}) \end{pmatrix}$$

We know that for each $n \in \mathbb{N}$, ϕ_n is a bounded linear operator, and so we define the *complete norm* of ϕ by

$$\left\|\phi\right\|_{cb} := \sup_{n\in\mathbb{N}} \left\|\phi_n\right\|$$

We say that ϕ is *completely bounded* if $\left\|\phi\right\|_{cb} < \infty$, and ϕ is *completely contractive* if $\left\|\phi\right\|_{cb} \leq 1$.

Arveson proved, amongst other things, that T has a rational ∂X-dilation, if and only if π is completely contractive, so the rational dilation conjecture can be rephrased as:

Conjecture. *The homomorphism $\pi : \mathcal{R}(X) \to B(K)$ such that $\pi(p/q) = p(T)\,q(T)^{-1}$ is contractive if and only if it's completely contractive.*

1.4. **Multiply connected domains.** We say a compact set $X \subseteq \mathbb{C}$ is an n-holed domain if its boundary, ∂X, has $n + 1$ components; this is kinda unintuitive, but if we look at Figure 1.1 on the facing page, we can see that this is really just the same as saying X has n holes in it.

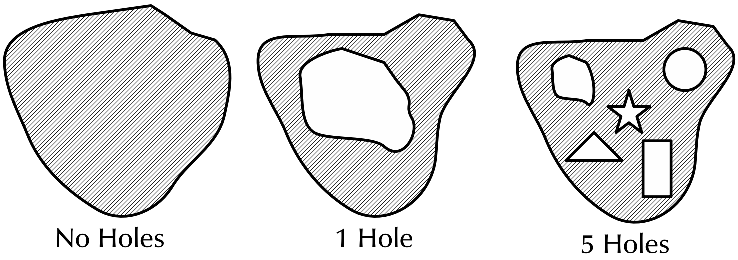

FIGURE 1.1. Planar domains with various numbers of holes

Agler showed that if X is a 1-holed domain with smooth boundary, then the rational dilation conjecture holds ([Agl85]). However, later work showed that the rational dilation conjecture doesn't hold on 2-holed domains: Agler, Harland and Raphael ([AHR08]) found an example of a 2-holed domain X and an operator T (in fact, their operator was just a 4×4 matrix), where X is a spectral set for T, but T doesn't have a ∂X-dilation. Dritschel and McCullough then went on to show that if we have any 2-holed domain X, with smooth boundary, then there exists an operator T, such that X is a spectral set for T, but T does not have a rational ∂X-dilation ([DM05]).

It's suspected that that the rational dilation conjecture doesn't hold on any 2-or-more-holed domain, but Dritschel and McCullough's proof doesn't easily generalise to all n-holed domains. However, I've been able to extend their proof to 2-or-more-holed domains with certain useful symmetries ([Pic08]). I'll be talking some more about these symmetries in the second part of this talk, but for now, all I'm going to say is that all 2-holed domains have these symmetries, so Dritschel and McCullough's result can, in a sense, be seen as a consequence of my result.

1.5. **Operators.** Something I should quickly mention before I go into the second part of the talk, is that not all of the research in this area has focused on the space X. There's also been some research into what kinds of operators will work too. For starters Misra, Paulsen and Agler, between them, showed that if our operator, T, is a 2×2 matrix, then rational dilation holds – so no matter how badly behaved X is, if T is a 2×2 matrix, and X is a spectral set for T, then T has a rational ∂X-dilation. On the other hand, as I mentioned before, Agler, Harland and Raphael's couterexample is a 4×4 matrix, so clearly rational dilation can't hold on all 4×4 matrices. At the moment, we don't know about 3×3 matrices, although in [AHR08], the authors hypothesise that rational dilation *does* hold for all 3×3 matrices, based on numerical evidence, although – to the best of my knowledge – there's no rigourous proof. You can find out more about this approach in [AHR08].

Our long term aim in all of this is to find a simple description of which operators have a rational ∂X-dilation, but I think we're quite a long way from being able to do this.

2. The counterexample

Now, although this talk is about the result in [Pic08], I'm taking a different approach to the one I take there. The reason for this is that I'd like to highlight some of the connections to other areas, including some areas of active research, which means taking a less rigourous approach to some things.

2.1. **Schur Agler Classes.** I'm going to discuss a seemingly unrelated result from another paper by Dritschel and McCullough ([DM07]). This paper gives various characterisations of scalar valued holomorphic functions, all in terms of so called "test functions" – scalar functions on a set X, with modulus at most 1 on X.

A set of test functions – which I usually call Θ – is essentially a microcosm for the set of all holomorphic functions. We can find the set $\mathcal{K}_\Theta$ of all kernels $k : X \times X \to \mathbb{C}$ that make

$$X \times X \ni (x, y) \to (1 - \psi(x)\overline{\psi(y)})k(x, y)$$

into a positive kernel, for all $\psi \in \Theta$.

We say a function $\varphi : X \to \mathbb{C}$ is holomorphic with respect to Θ (denoted $\varphi \in H^\infty(\mathcal{K}_\Theta)$) if there exists some $\rho > 0$, such that if $k \in \mathcal{K}_\Theta$, then

$$X \times X \ni (x, y) \to (\rho^2 - \varphi(x)\overline{\varphi(y)})k(x, y)$$

is also a positive kernel, so φ behaves like our test functions in some sense. We can then find a norm, $\|\varphi\|_{H^\infty(\mathcal{K}_\Theta)}$, which is the infimum over all possible ρ.

The main result of [DM07] is the following:

Theorem. *If Θ is a collection of test functions, then the following are equivalent:*

(1) $\varphi \in H^\infty(\mathcal{K}_\Theta)$ *and* $\|\varphi\|_{H^\infty(\mathcal{K}_\Theta)} \leq 1$
(2) *For each finite set $F \subseteq X$, there exists a positive kernel $\Gamma : F \times F \to C_b(\Theta)^*$ such that for all $x, y \in F$,*

$$1 - \varphi(x)\overline{\varphi(y)} = \Gamma(x, y)(1 - E(x)\overline{E(y)})$$

where $E(x)$ is the mapping in $C_b(\Theta)$ that takes $\psi \in \Theta$ to $\psi(x)$.
(3) *There exists a positive kernel $\Gamma : X \times X \to C_b(\Theta)^*$ such that for all $x, y \in X$,*

$$1 - \varphi(x)\overline{\varphi(y)} = \Gamma(x, y)(1 - E(x)\overline{E(y)})$$

(4) *There is a colligation Σ such that $\varphi = W_\Sigma$ (the definitions are not reproduced here).*

(5) *For every representation* π *of* $H^\infty(\mathcal{K}_\Theta)$ *such that* $\|\pi(\psi)\| < 1$ *for all* $\psi \in \Theta$, $\|\pi(\varphi)\| \leq 1$

(6) *For every weakly continuous representation* π *of* $H^\infty(\mathcal{K}_\Theta)$ *such that* $\|\pi(\psi)\| \leq 1$ *for all* $\psi \in \Theta$, $\|\pi(\varphi)\| \leq 1$.

For this talk, we're only interested in (1), (6), and to a lesser extent (3), although they all have roles to play.

Now, Dritschel and McCullough's paper only deals with scalar-valued holomorphic functions, but for the purpose of this talk I'm going to make an assumption I don't make in the paper; I'm going to assume this result also holds for matrix-valued holomorphic functions.

One question we can then ask is whether the set of test functions we use for the scalar-valued holomorphic functions will work for the matrix-valued holomorphic functions. Since these test functions are scalar-valued, it doesn't matter whether we think of them as being scalar-valued or matrix-valued – if they need to be matrix-valued we can just take $\widehat{\Theta} = \{\psi I \mid \psi \in \Theta\}$. In fact, what I hope to put across to you today, is that the set of test functions we use has to be properly matrix-valued; scalar-valued just won't do.

The reason this relates to rational dilation, is part 6 of the theorem. If rational dilation fails for some operator T with spectral set X, then since X is a spectral set for T, $\|\psi(T)\| \leq 1$ for all $\psi \in \Theta$. However, we know that the homomorphism π taking holomorphic scalar-valued functions $f \to f(T)$ is not completely contractive, so there must be some π_n which is not contractive, so there must be some holomorphic matrix-valued function φ such that $\|\varphi\| \leq 1$ but

$$\|\pi_n(\varphi)\| = \|\varphi(T)\| = \left\| \begin{pmatrix} \varphi_{11}(T) & \cdots & \varphi_{1n}(T) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(T) & \cdots & \varphi_{nn}(T) \end{pmatrix} \right\| > 1$$

This is a problem, as this means that $\|\varphi\| \leq 1$ but $\|\varphi\|_{H^\infty(\mathcal{K}_\Theta)} > 1$, so Θ is not the right set of test functions, as it gives different norms.

Also, it's a little less obvious, but we can take the last argument the other way – if we have a matrix-valued holomorphic function φ such that $\|\varphi\| \leq 1 < \|\varphi\|_{H^\infty(\mathcal{K}_\Theta)}$, then this gives us a counterexample to rational dilation. This is the approach I'm going to take; I'm going to construct a 2×2 matrix-valued holomorphic function with $\|\varphi\| \leq 1 < \|\varphi\|_{H^\infty(\mathcal{K}_\Theta)}$.

2.2. **Symmetric Domains.** For the construction I use, X needs to have certain properties; it needs to have analytic boundary curves, and its interior needs to be conformally equivalent to a domain which is symmetric about the real axis. Now, this turns out to be equivalent to some other useful properties, such as having a hyperelliptic Schottky double ([Bar75]). Having said that though, it's not a major loss of generality to just assume X *is*

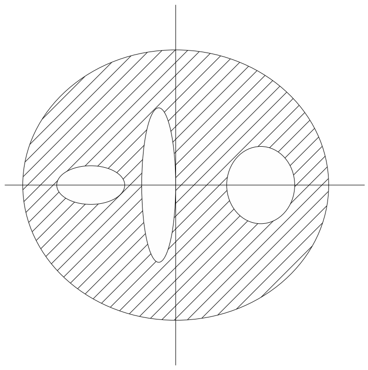

FIGURE 2.1. An example of a symmetric domain

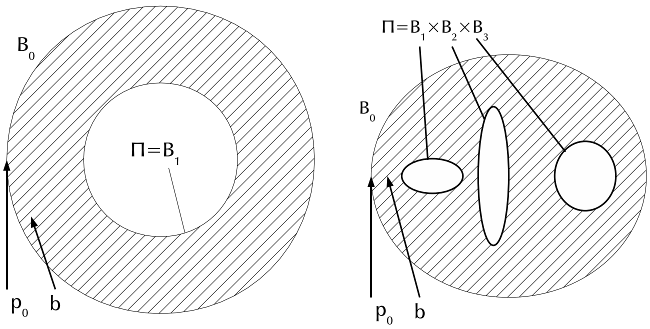

FIGURE 2.2. A map of the annulus, and a symmetric domain

symmetric about the real axis – rather than just conformally equivalent to something that is – and it makes the calculations a lot simpler, so I'm going to make this assumption. See figure 2.1 for an example of such a domain.

The next thing we'll need is a good set of test functions for our domain X. Dritschel and McCullough give of a minimal set of test functions for the annulus $\mathbb{A}$ in [DM07]. Looking at Figure 2.2, this set is given by

$$\Theta_{\mathbb{A}} := \left\{ \psi_p : p \in \Pi \right\}$$

where ψ_p is the (unique) inner function which takes the value 1 exactly twice – at p_0 and p – and is equal to zero at b. We can define a set of test functions on X in a similar way, by

$$\Theta_X := \left\{ \psi_p : p = (p_1, \dots, p_n) \in \Pi \right\}$$

where ψ_p is the unique inner function that takes the value 1 exactly $n + 1$ times – at $p_0, p_1, \dots, p_n$ – and is equal to zero at b. I should say that the fact that b and p_0 are on the real line is important, although I won't go into why.

Now, there's another, equivalent way to construct Θ_X, which will prove useful later. We know, by solving the Dirichlet problem, that if we've got a measure μ on ∂X, there is some harmonic function h_μ on the interior of

X with boundary values given by μ. We can also show that for any $p \in \Pi$, there is a unique measure μ_p with point masses at $p_0, p_1, \ldots, p_n$, such that h_{μ_p} is the real part of an analytic function g_p, and $h_{\mu_p}(b) = 1$. If we then require that $g_p(b) = 1$, then g_p is uniquely defined. Since g_p only takes values on the right half-plane, and $g_p(b) = 1$, it's a simple Möbius transform to turn g_p into ψ_p, in fact the Möbius transform doesn't even depend on p.

The real reason this equivalent construction is interesting though, is h_{μ_p}. We can break down h_{μ_p} into $h_p^0 + h_p^1 + \cdots + h_p^n$, where h_p^i is zero everywhere on ∂X except for a point mass at p_i, so h_p^1 just has a point mass at p_1, h_p^2 just has a point mass at p_2, etc. I'll be using these h_p^is to construct our counterexample.

2.3. **The counterexample.** So, how do we construct our counterexample? First, let's look at a nice holomorphic function, if we define

$$H_p = \begin{pmatrix} h_p^0 & 0 \\ 0 & h_{\overline{p}}^0 \end{pmatrix} + \begin{pmatrix} h_p^1 & 0 \\ 0 & h_{\overline{p}}^1 \end{pmatrix} + \cdots + \begin{pmatrix} h_p^n & 0 \\ 0 & h_{\overline{p}}^n \end{pmatrix} = \begin{pmatrix} h_{\mu_p} & 0 \\ 0 & h_{\mu_{\overline{p}}} \end{pmatrix}$$

then clearly there's nothing wrong with this. We can take it as the real part of a matrix-valued holomorphic function G_p, which works out as $G_p = \begin{pmatrix} g_p & 0 \\ 0 & g_{\overline{p}} \end{pmatrix}$, and we can then Möbius transform it to $\Psi_p = \begin{pmatrix} \psi_p & 0 \\ 0 & \psi_{\overline{p}} \end{pmatrix}$, which is a perfectly well behaved holomorphic function, and can be represented as in (3).

So, how do we break it? We do something like this:

$$\widetilde{H_p} = \begin{pmatrix} h_p^0 & 0 \\ 0 & h_{\overline{p}}^0 \end{pmatrix} + U_1^* \begin{pmatrix} h_p^1 & 0 \\ 0 & h_{\overline{p}}^1 \end{pmatrix} U_1 + \cdots + U_n^* \begin{pmatrix} h_p^n & 0 \\ 0 & h_{\overline{p}}^n \end{pmatrix} U_n$$

for some unitary matrices $U_1, \ldots, U_n$. Although this is kinda messy, we can still take it as the real part of a matrix-valued holomorphic function, and we can still Möbius transform it to an inner function $\widetilde{\Psi_p}$, so $\left\|\widetilde{\Psi_p}\right\| \leq 1$. However, we can then show, via several pages of calculations, that if $\left\|\widetilde{\Psi_p}\right\|_{H^\infty(\mathcal{K}_\Theta)} \leq 1$, and we choose p, b, and our unitaries carefully enough, then $\widetilde{\Psi_p}$ should be diagonalisable, so it should be of the form $\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$, which it isn't, so we're forced to conclude that $\left\|\widetilde{\Psi_p}\right\| \leq 1 < \left\|\widetilde{\Psi_p}\right\|_{H^\infty(\mathcal{K}_\Theta)}$. Therefore, rational dilation fails on X, for any suitably symmetric X.

References

[Agl85] Jim Agler, *Rational dilation on an annulus*, The Annals of Mathematics **121** (1985), no. 3, 537–563.

[AHR08] Jim Agler, John Harland, and Benjamin J. Raphael, *Classical function theory, operator dilation theory, and machine computation on multiply-connected domains*, Memoirs of the American Mathematical Society **191** (2008), no. 892, viii+159.

[Bar75] William H. Barker, *Plane domains with hyperelliptic double*, Ph.D. thesis, Stanford, 1975.

[DM05] Michael A. Dritschel and Scott McCullough, *The failure of rational dilation on a triply connected domain*, Journal of The American Mathematical Society **18** (2005), no. 4, 873–918.

[DM07] ______, *Test functions, kernels, realizations and interpolation*, Operator Theory, Structured Matrices and Dilations: Tiberiu Constantinescu Memorial Volume, pp. 153–179, Theta Foundation, Bucharest, 2007.

[Pau02] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, Cambridge, 2002.

[Pic08] James Pickering, *Counterexamples to rational dilation on symmetric multiply connected domains*, Complex Analysis and Operator Theory (2008), Online edition, yet to appear in print.

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