Test Function Realisations and Agler-Herglotz Representations

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22nd September 2009

Introduction

Test function realisations are useful characterisations of function algebras.

These were first developed by Jim Agler, in the late 1980's. They were used to solve the Nevanlinna-Pick interpolation problem on the bidisc.

To do this, Agler gave a realisation for $H^\infty(\mathbb{D}^2)$.

These techniques have since been applied to other function algebras, to solve other problems in function theory and operator theory.

These test functions are unrelated to the test functions in distribution theory

Realisations are often based on generalisations of Herglotz' Representation Theorem – so called **Agler-Herglotz representations**.

If we can find an Agler-Herglotz representation for a particular function algebra, we can often use that to find a realisation for that algebra.

Interpolation on the Bidisc

If we have n points $x_1, \ldots, x_n$ in $\mathbb{D}^2$ (the bidisc), and another n points $y_1, \ldots, y_n$ in $\mathbb{D}$ (the disc), when do we have a holomorphic function

$$f : \mathbb{D}^2 \to \mathbb{D}$$
$$f : x_1 \to y_1$$
$$\vdots$$
$$f : x_n \to y_n?$$

Equivalently, we could ask: When do we have a function $f \in H^\infty(\mathbb{D}^2)$ with $\|f\| \leq 1$ and $f(x_i) = y_i$?

The Solution

Agler showed that for any contractive holomorphic function $f : \mathbb{D}^2 \to \mathbb{D}$, there are two positive definite kernels

$$k_1, k_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$$

such that

$$1 - f(z)\overline{f(w)} = (1 - z_1\overline{w_1})k_1(z, w) + (1 - z_2\overline{w_2})k_2(z, w)$$

for any $z = (z_1, z_2) \in \mathbb{D}^2$, and $w = (w_1, w_2) \in \mathbb{D}^2$.

We will say that k_1 and k_2 **realise** f.

In fact, if a formula like this holds at finitely many points in $\mathbb{D}^2$, then we can extend this to the whole of $\mathbb{D}^2$. That is to say, if we have a finite set $F \subset \mathbb{D}^2$ and two kernels

$$k_1, k_2 : F \times F \to \mathbb{C}$$

such that

$$1 - f(z)\overline{f(w)} = (1 - z_1\overline{w_1})k_1(z,w) + (1 - z_2\overline{w_2})k_2(z,w)$$

for any $z, w \in F$, then we can extend our kernels to the whole of $\mathbb{D}^2 \times \mathbb{D}^2$, and our function f becomes a holomorphic function on $\mathbb{D}^2$.

If our function f is only defined on a finite set F, but we have kernels k_1 and k_2 that realise f on F, then we can extend f to the whole of $\mathbb{D}^2$.

This solves the interpolation problem; we have a holomorphic function function $\|f\| \leq 1$ taking x_i to y_i whenever we have kernels k_1 and k_2 that realise f.

Test Function Realisations

Agler's result has all the basic features of a **test function realisation**. We can write $1 - f(z)\overline{f(w)}$ as a sum or integral of things that look like

$$\left(1 - \psi(z)\overline{\psi(w)}\right) k_\psi(z, w)$$

Compare this to

$$1 - f(z)\overline{f(w)} = (1 - z_1\overline{w_1})k_1(z, w) + (1 - z_2\overline{w_2})k_2(z, w)$$

We are summing over a set of functions $\Psi = \{z_1, z_2\}$, which are, in some sense, building blocks for all the other functions. We can build all holomorphic functions in $H^\infty(\mathbb{D}^2)$ in this way.

Another Example

Dritschel and McCullough developed a similar realisation for $H^\infty(\mathbb{A})$ – the holomorphic functions on the annulus. Their set of test functions was parameterised by the circle, so

$$\Psi = \{\psi_\theta : \theta \in [0, 2\pi)\} \ .$$

They showed that for any function $f \in H^\infty(\mathbb{A})$ with $\|f\|_{H^\infty(\mathbb{A})} \leq 1$, there are kernels k_θ and a measure μ, such that:

$$1 - f(z)\overline{f(w)} = \int_0^{2\pi} \left(1 - \psi_\theta(z)\overline{\psi_\theta(w)}\right) k_\theta(z, w)\, d\mu(\theta)$$

Dritschel and McCullough's realisation used a result similar to Herglotz representation theorem – a so called **Agler-Herglotz representation**.

Classical Herglotz Representation Theorem

Theorem

f is holomorphic function on $\mathbb{D}$ with $\Re f \geq 0$ and $f(0) > 0$, if and only if there exists a measure μ on $\mathbb{T}$ with

$$f(z) = \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} d\mu(\lambda)$$

The classical Herglotz representation theorem allows us to break up functions on the disk into simple "**building block**" functions of the same type.

We have a "space" of these building block functions ($\mathbb{T}$, in this case), and our function f is an integral over this space, with respect to μ.

Classical Herglotz Representations

Theorem (repeated)

f is holomorphic function on $\mathbb{D}$ with $\Re f \geq 0$ and $f(0) > 0$, if and only if there exists a measure μ on $\mathbb{T}$ with

$$f(z) = \int_{\mathbb{T}} \frac{\lambda + z}{\lambda - z} d\mu(\lambda)$$

Since our function f is an integral over $\mathbb{T}$ with respect to μ, functions like f correspond to measures like μ. We're representing f by the measure μ

For this talk, we'll use H^+ (analogously to H^∞) to denote the space of functions that satisfy the conditions.

Generalisation

We want to generalise the Herglotz representation theorem to other settings. For example, we could look at functions on arbitrary planar domains $X \subseteq \mathbb{C}$, rather than $\mathbb{D}$.

We want to represent functions in $H^+(X)$ by other, simple, "building block" functions from $H^+(X)$ – we want to use functions **of the same type**

To do this, it might help to understand where our existing building block functions come from.

Building Blocks

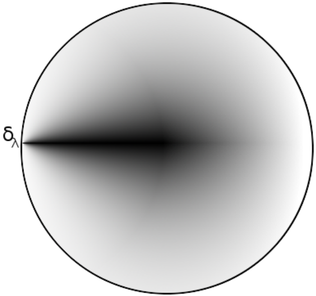
The Herglotz representation was

$$f(z) = \int_{\mathbb{T}} \underbrace{\frac{\lambda + z}{\lambda - z}}_{h_\lambda(z)} d\mu(\lambda)$$

This formula comes about by thinking about boundary values of **harmonic functions**.

The real part of f is a harmonic function on $\mathbb{D}$ and the real part of our function $h_\lambda(z)$ is the Poisson kernel on $\mathbb{D}$.

Harmonic Functions



In some sense, $\Re f$ is the harmonic function which is μ **on the boundary**.

In this regard $\Re h_\lambda$ is the harmonic function which is δ_λ (a point mass at λ) on the boundary.

$$f(z) = \int_{\mathbb{T}} \underbrace{\frac{\lambda + z}{\lambda - z}}_{h_\lambda(z)} d\mu(\lambda)$$

To find simple building block functions, we need to find simple **building block measures**.

Multiply Connected Domains

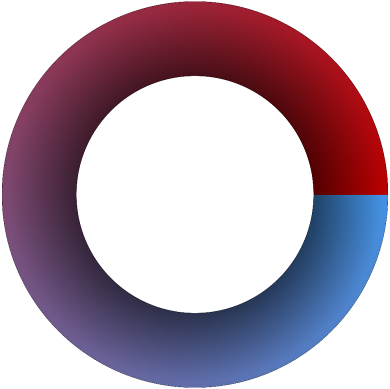
If our domain is some arbitrary planar domain X, rather than $\mathbb{D}$, we can still translate measures into **harmonic** functions.

However, we want a representation for **holomorphic** functions; we want to break holomorphic functions up into simple building block functions $h_\lambda(z)$.

If X is not simply connected, not every harmonic function corresponds to a holomorphic function.

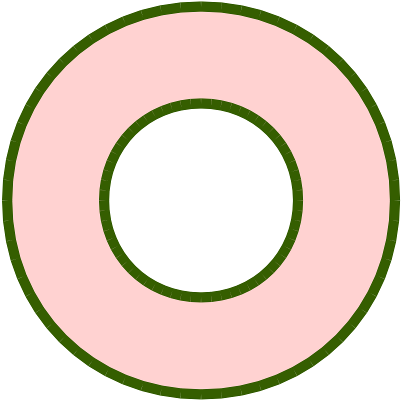
We want simple building block measures μ_λ, for simple building block functions $h_\lambda(z)$.

An illustration on the Annulus

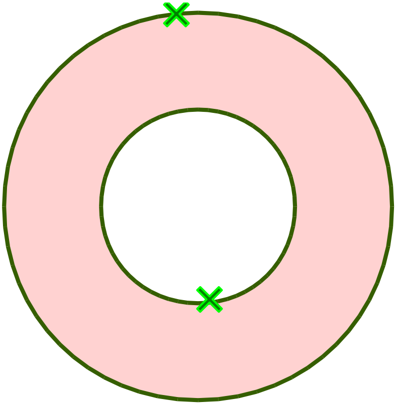


On the annulus $\mathbb{A}$, $\log|z|$ is a harmonic function, but is not the real part of a holomorphic function.

If it were the real part of a holomorphic function, it would be the real part of $\log(z)$. $\log(z)$ increases by $2\pi i$ along closed paths that wind round the origin, so $\log(z)$ is not a well defined function.



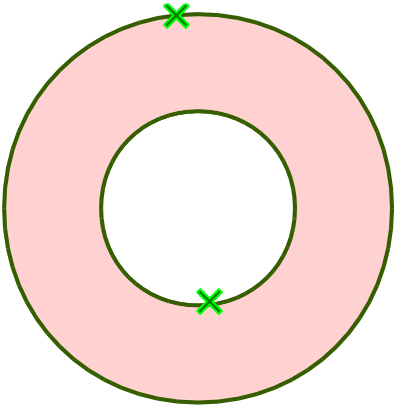
We want our measure μ to correspond to a well-defined holomorphic function f. A necessary condition for this, is that μ must be supported somewhere on the inner boundary of $\mathbb{A}$, and somewhere on the outer boundary.



We want to find simple "building block" measures, that we can use to build all measures of this type.

The simplest possible measure is supported at exactly two points, one on the inner boundary, one on the outer boundary.

We call one of these building block measures λ, and the corresponding function h_λ.



λ must be supported at two points λ_1, λ_2, and we can describe h_λ by the pair (λ_1, λ_2).

Since both boundaries of $\mathbb{A}$ are circles, we can think of these points as being in the torus $\mathbb{T}^2$

We have a “space” of functions:

$$\left\{h_\lambda : \lambda \in \mathbb{T}^2\right\} \cong \mathbb{T}^2$$

The Agler-Herglotz Representation

This leads to an Agler-Herglotz representation:

Every function $f \in H^+(\mathbb{A})$ can be written as

$$f(z) = \int_{\mathbb{T}^2} h_\lambda(z) d\mu(\lambda)$$

for some measure μ on $\mathbb{T}^2$.

Constrained Interpolation

Various authors have worked on the interpolation problem for

$$H_1^\infty := \{f \in H^\infty : f'(0) = 0\}$$

We would like to find a test function realisation for H_1^∞.

As before, this is accomplished by finding an Agler-Herglotz representation for H_1^+.

An Agler-Herglotz representation represents functions in H_1^+ by functions **of the same type**.

For this, we need to know which measures μ on $\mathbb{T}$ (the boundary of $\mathbb{D}$) correspond to functions f in H_1^+.

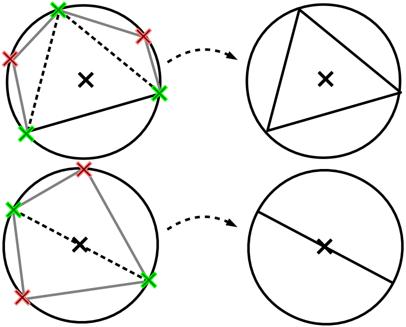
We can show that $f'(0) = 0$ if and only if

$$\int_{\mathbb{T}} z d\mu(z) = 0$$

We can safely assume that μ is a probability measure, so this condition says that the expectation of μ is zero.

We call μ a **zero-mean probability measure**.

Simple Measures



We want to find simple “building block” zero-mean probability measures.

Finitely supported measures are relatively simple, but can often be simplified further.

If we repeatedly remove unnecessary points, we can reduce our measures to these two possibilities.

These simple measures are also **extreme points** in the sense of the Krein-Milman Theorem (we actually use Choquet's Theorem), so can be used to construct all zero-mean measures.

If we consider the "space" Θ of simple zero-mean probability measures (under the weak-$*$ topology), then we have an Agler-Herglotz representation for functions $f \in H_1^+$:

$$f(z) = \int_\Theta h_\vartheta(z) d\mu(\vartheta)$$

for some measure μ on Θ

Closing Remarks

Once we have an Agler-Herglotz representation for a space, it’s fairly easy to find a test function realisation.

Test function realisations, and Agler-Herglotz representations have proved to be useful tools. They were instumental in disproving the rational dilation conjecture, and (I’m told) have applications to control theory.

Further Reading

- My work: `http://www.jamespic.me.uk`
- **A Constrained Nevanlinna-Pick Interpolation Problem**, Davidson, Paulsen, Raghupathi and Singh
- **Test Functions, Kernels, Realizations and Interpolation**, Dritschel and McCullough
- **Pick Interpolation and Hilbert Function Spaces**, Agler and McCarthy
- **Classical function theory, operator dilation theory, and machine computation on multiply-connected domains**, Agler, Harland and Raphael