

COUNTEREXAMPLES TO RATIONAL DILATION ON SYMMETRIC MULTIPLY CONNECTED DOMAINS

JAMES PICKERING

ABSTRACT. We show that if R is a compact domain in the complex plane with two or more holes and an anticonformal involution onto itself (or equivalently a hyperelliptic Schottky double), then there is an operator T which has R as a spectral set, but does not dilate to a normal operator with spectrum on the boundary of R .

0.1. Definitions. Let X be a compact, path connected subset of \mathbb{C} , with interior R , and *analytic boundary* B composed of $n + 1$ disjoint curves, B_0, \dots, B_n , where $n \geq 2$. By analytic boundary, we mean that for each boundary curve B_i there is some biholomorphic map ϕ_i on a neighbourhood U_i of X which maps B_i to the unit circle \mathbb{T} . By convention B_0 is the outer boundary. We write $\Pi = B_0 \times \dots \times B_n$.

We say a Riemann surface Y is *hyperelliptic* if there is a meromorphic function with two poles on Y (see [FK92]). We say R is symmetric if there exists some anticonformal involution ω on R with $2n + 2$ fixed points on B . We say a domain in $\mathbb{C} \cup \{\infty\}$ (that is, the Riemann sphere S^2) is a *real slit domain* if its complement is a finite union of closed intervals in $\mathbb{R} \cup \{\infty\}$.

We define $\mathcal{R}(X) \subseteq C(X)$ as the space of all rational functions that are continuous on X . The definitions of contractivity and complete contractivity are the usual definitions, and can be found in [Pau02].

0.2. Introduction. A key problem that this paper deals with is the *rational dilation conjecture*, which is as follows.

Conjecture 0.1. *If $X \subseteq \mathbb{C}$ is a compact domain, $T \in \mathcal{B}(H)$ is a Hilbert space operator with $\sigma(T) \subseteq X$ and $\|f(T)\| \leq 1$ for all $f \in \mathcal{R}(X)$, then there is some normal operator $N \in \mathcal{B}(K)$, $K \supseteq H$, such that $\sigma(N) \subseteq B (= \partial X)$, and $f(T) = P_H N|_H$.*

A classical result of Sz.-Nagy shows that the rational dilation conjecture holds if X is the unit disc. A generalisation by Berger, Foias and Lebow

Date: Submitted on 29th June 2009.

2000 Mathematics Subject Classification. Primary 47A20; Secondary 47A25, 47A48, 46E22, 30C20, 30H05, 30F10, 30E05.

Key words and phrases. Rational Dilation, Hyperelliptic Riemann Surfaces, Nevanlinna-Pick Interpolation.

This paper is based on work contributing towards the author's PhD thesis at the University of Newcastle-upon-Tyne, under the supervision of Michael Dritschel. This work is funded by the Engineering and Physical Sciences Research Council.

shows this holds for any simply connected domain (see [Pau02]). A result by Agler (see [Agl85]) shows that rational dilation also holds if X has one hole – such as in an annulus. However, subsequent work has shown that rational dilation fails on every two-holed domain with analytic boundary (see [DM05], and [AHR08]).

The aim of this paper is to prove the following, which by a result of Arveson (see [Pau02, Cor. 7.8]), is equivalent to showing that the rational dilation conjecture does not hold on any symmetric, two-or-more-holed domain.

Theorem 0.2. *If X is a symmetric domain in \mathbb{C} , with $2 \leq n < \infty$ holes, there is an operator $T \in \mathcal{B}(H)$, for some Hilbert space H , such that the homomorphism $\pi : \mathcal{R}(X) \rightarrow \mathcal{B}(H)$ with $\pi(p/q) = p(T) \cdot q(T)^{-1}$ is contractive, but not completely contractive.*

Proof Outline. First, we let C define the cone generated by

$$\left\{ H(z) \left[1 - \psi(z) \overline{\psi(w)} \right] H(w)^* : \psi \in \mathcal{BH}(X), H \in M_2(\mathbb{H}(X)) \right\},$$

where $\mathcal{BH}(X)$ is the unit ball of the space of functions analytic in a neighbourhood of X , under the supremum norm, and $M_2(\mathbb{H}(X))$ is the space of 2×2 matrix valued functions analytic in a neighbourhood of X . For $F \in M_2(\mathbb{H}(X))$, we set

$$\rho_F = \sup \left\{ \rho > 0 : I - \rho^2 F(z) F(w)^* \in C \right\}.$$

We show that there exists a function F which is unitary valued on B (we say F is *inner*), but such that $\rho_F < 1$. We show that such a function generates a counter-example of the type needed. To show that such a function exists, we show that if F is inner, $\rho_F = 1$ ($\|F\| = 1$ by the max modulus principle, so $\rho_F \leq 1$), and the zeroes of F are “well behaved”, then F can be diagonalised. We go on to show that there is a non-diagonalisable inner function F , with well behaved zeroes, which must therefore have $\rho_F < 1$, so must be a counter-example. \square

1. SYMMETRIES

Details of the ideas discussed below can be found in [Bar75]. A less detailed (but more widely available) presentation can be found in [Bar77].

Theorem 1.1. *Let $R \subseteq \mathbb{C}$ have $n + 1$ analytic boundary curves, $B_0, \dots, B_n \subseteq B$, with $n \geq 2$, and let Y be its Schottky double. The following are equivalent:*

- (1) Y is hyperelliptic;
- (2) R is symmetric;
- (3) R is conformally equivalent to a real slit domain Ξ .

The proof can be found in [Bar75], but we will briefly discuss the constructions involved. We know from [FK92, III.7.9] that Y is hyperelliptic

if and only if there is a conformal involution $\iota : Y \rightarrow Y$ with $2n + 2$ fixed points. We find that ι is given by

$$\iota(x) = \begin{cases} J \circ \omega(x) & x \in R \\ \omega(x) & x \in B \\ \omega \circ J(x) & x \in J(R) \end{cases},$$

where J is the ‘‘mirror’’ function on Y .

Also, if $\zeta : \Xi \rightarrow R$ is the conformal mapping from part 3, we have that $\omega(\zeta(\xi)) = \zeta(\bar{\xi})$.

Definition 1.2. We define the *fixed point set* of our symmetric domain R as

$$\mathbb{X} := \{x \in R : x = \omega(x)\}.$$

Remark 1.3. In view of Theorem 1.1 on the facing page, it makes sense to relabel the components of B . We can see that \mathbb{X} must be the image of $\mathbb{R} \cap \Xi$ under ζ , so must consist of a finite collection of paths running between fixed points of B . We choose one of the two fixed points of B_0 , and call it p_0^- . We follow \mathbb{X} from p_0^- to another B_i which we relabel B_1 ; we call the fixed point we landed at p_1^+ . Label the other fixed point in B_1 as p_1^- , and repeat, until we reach p_0^+ . The section of \mathbb{X} from p_i^- to p_{i+1}^+ , we call \mathbb{X}_i .

Proposition 1.4. *If a meromorphic function on Y has n or fewer poles, and all of these poles lie in $R \cup B$, then all of these poles must lie on B .*

Proof. Suppose f has n or fewer poles. Then $f \circ \iota$ also has n or fewer poles, so $f - f \circ \iota$ has $2n$ or fewer poles. However, if x is a fixed point of ι , $f(x) - f \circ \iota(x) = 0$, and since ι has $2n + 2$ fixed points, $f - f \circ \iota$ has at least $2n + 2$ zeroes. This is only possible if $f - f \circ \iota \equiv 0$, so if x is a pole of f , then $\iota(x)$ is a pole of f , which is a contradiction unless $x \in B$. \square

2. INNER FUNCTIONS

Many of the ideas found in this section can also be found in [AHR08] and [DM05].

Results in this section often require us to choose a point $b \in R$. Usually, b will be determined by the particular application, but in this section we make no requirements on the choice of b .

2.1. Harmonic and Analytic Functions. If ω_b is harmonic measure at b , and s is arc length measure, by an argument like the one in [DM05], we can find a Poisson kernel $\mathbb{P} : R \times B \rightarrow \mathbb{R}$ such that for h harmonic on R and continuous on B ,

$$h(w) = \int_B h(z) \mathbb{P}(w, z) ds(z).$$

Equivalently, \mathbb{P} is given by the Radon-Nikodym derivative

$$\mathbb{P}(w, \cdot) = \frac{d\omega_w}{ds}.$$

We know that \mathbb{P} is harmonic in R at each point in B , and that for any positive h harmonic on R , and continuous on X there exists some positive measure μ on B such that

$$h(w) = \int_B \mathbb{P}(w, z) d\mu(z).$$

Conversely, given a positive measure μ on B , this formula defines a positive harmonic function.

We let h_j denote the solution to the Dirichlet problem which is 1 on B_j and 0 on B_i , where $i \neq j$. We can see that this corresponds to the arc length measure on B_j .

We define $Q_j : B \rightarrow \mathbb{R}$ as the outward normal derivative of h_j , and define the *periods* of h by

$$P_j(h) = \int_B Q_j d\mu.$$

It should be clear that h is the real part of an analytic function if and only if $P_j(h) = 0$ for $j = 0, 1, \dots, n$.

Lemma 2.1. *The functions Q_j have no zeroes on B . Moreover, $Q_j > 0$ on B_j and $Q_j < 0$ on B_l for $l \neq j$.*

Proof. As X has analytic boundary, we can assume without loss of generality that $B_0 = \mathbb{T}$. We know that h_j takes its minimum and maximum on its boundary. Since h_j equals one on B_j , and zero on B_l if $l \neq j$, these must be its maximum and minimum respectively, so h_j is non-decreasing towards B_j , and non-increasing towards B_l , so $Q_j \geq 0$ on B_j and $Q_j \leq 0$ on B_l .

We can see by the above argument that we only need show that $Q_j \neq 0$. We let R' be the reflection of R about B_0 (which we are assuming is the unit circle). We can extend h_j to a harmonic function on $X \cup R'$ by setting

$$h_j(z) = -h_j(1/\bar{z})$$

on R' .

If Q_j had infinitely many zeroes on B_0 , then Q_j would be identically zero, so we suppose Q_j has finitely many zeroes on B_0 .

Suppose Q_j has a zero z , and a small, simply connected neighbourhood $N(z)$. By choosing $N(z)$ small enough, we can ensure that $N(z)$ contains no other zeroes. Clearly, h_j forms the real part of some holomorphic function f on $N(z)$. We know that $\partial h_j / \partial n = Q_j = 0$, and because h_j is constant on B_0 , we know that the tangential derivative of h_j , $\partial h_j / \partial t$, is also zero, so f has derivative zero at z , so f has a ramification of order at least two at z . We also know that f maps everything outside the unit disc to the left half

plane, and everything inside the unit disc to the right half plane, but clearly this is impossible, so Q_j cannot have a zero.

A similar argument holds for B_1, \dots, B_n . \square

Corollary 2.2. *If h is a non-zero positive harmonic function on R which is the real part of an analytic function, and h is represented in terms of a positive measure μ , then $\mu(B_j) > 0$ for each j .*

Proof. If $\mu(B_j) = 0$, then as $Q_j < 0$ on $B \setminus B_j$, $P_j(h) < 0$, a contradiction. Thus, $\mu(B_j) > 0$. \square

2.2. Some Matrix Algebra. We wish to show that at each $p \in \Pi$, the vector

$$V^n = \det \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ Q_1(p_0) & Q_1(p_1) & \cdots & Q_1(p_n) \\ Q_2(p_0) & Q_2(p_1) & \cdots & Q_2(p_n) \\ \vdots & \vdots & \ddots & \vdots \\ Q_n(p_0) & Q_n(p_1) & \cdots & Q_n(p_n) \end{pmatrix}$$

has only positive coordinates. It helps to note that in three dimensions

$$\mathbf{x} \times \mathbf{y} = \det \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix}.$$

It will also be helpful to write

$$V^n = \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_n \\ - & + & - & - & \cdots & - \\ - & - & + & - & \cdots & - \\ - & - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & - & \cdots & + \end{vmatrix},$$

noting that $Q_j(p_j) > 0$, and $Q_i(p_j) < 0$ for $i \neq j$. From here on, positive and negative quantities will simply be denoted by (+) and (−), respectively.

Lemma 2.3. *All sub-matrices of V^n of the form*

$$\begin{pmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \cdots & + \end{pmatrix}$$

have positive determinant.

Proof. We can assume, without loss of generality, that such matrices are of the form

$$\begin{pmatrix} Q_1(p_1) & Q_1(p_2) & \cdots & Q_1(p_k) \\ Q_2(p_1) & Q_2(p_2) & \cdots & Q_2(p_k) \\ \vdots & \vdots & \ddots & \vdots \\ Q_k(p_1) & Q_k(p_2) & \cdots & Q_k(p_k) \end{pmatrix} := A^T$$

by a simple relabelling of boundary curves. We note that

$$\sum_{j=0}^n h_j \equiv 1,$$

so in particular

$$\sum_{j=0}^n Q_j(x) = 0$$

for all $x \in B$. So, if $1 \leq i \leq k$, then

$$\sum_{j=1}^k Q_j(p_i) = - \left(Q_0(p_i) + \sum_{j=k+1}^n Q_j(p_i) \right) > 0.$$

We now apply Gershgorin's circle theorem. Since $A_{ij} = Q_j(p_i)$, the eigenvalues of A are in the set

$$S := \bigcup_{i=1}^N D \left(\sum_{\substack{j=1 \\ j \neq i}}^n A_{ij}, A_{ii} \right) := \bigcup_{i=1}^N S_i,$$

where $D(\epsilon, x) \subseteq \mathbb{C}$ is the ball centred at x of radius ϵ . Now, if $\lambda \in S_i$, then $|\lambda - A_{ii}| < \sum_{j \neq i} A_{ij}$, so in particular

$$\Re(\lambda) > A_{ii} - \sum_{j \neq i} |A_{ij}| = A_{ii} + \sum_{j \neq i} A_{ij} = \sum_{j=1}^n A_{ij} > 0.$$

Now, all terms in the matrix A are real, so if λ is an eigenvalue of A , then either $\lambda > 0$, or $\bar{\lambda}$ is also an eigenvalue. We know that the determinant of a matrix is given by the product of its eigenvalues, counting multiplicity. Therefore, the determinant of A is a product of positive reals, and terms of the form $\lambda \bar{\lambda} = |\lambda|^2$, which are also positive and real, so $\det(A)$ is positive, so $\det(A^T)$ is positive. \square

Lemma 2.4. V^n has only positive coefficients.

Proof. We define

$$d_i^n = \left| \begin{array}{c|cccc} & \overbrace{\quad\quad\quad}^{i \times i} & & & \\ \hline - & + & - & \cdots & - \\ - & - & + & \ddots & \vdots \\ - & - & - & \ddots & - \\ \vdots & \vdots & \ddots & \ddots & + \\ - & \cdots & - & - & - \\ \hline & & & & + & - & - & \cdots & - \\ & & & & - & + & - & \ddots & - \\ & & - & & - & - & + & \ddots & \vdots \\ & & & & \vdots & \vdots & \ddots & \ddots & - \\ & & & & - & - & \cdots & - & + \end{array} \right|.$$

For our purposes, all that matters is the signs of the elements of this matrix, and that Lemma 2.3 on page 5 holds. Cyclically permuting the first i rows gives

$$d_i^n = (-1)^{i-1} \left| \begin{array}{ccccc} - & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & \vdots \\ - & \vdots & \ddots & \ddots & - \\ - & - & \cdots & - & + \end{array} \right| = (-1)^{i-1} d_1^n.$$

We can see that

$$V^n = \left| \begin{array}{cccccc} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_n \\ - & + & - & - & \cdots & - \\ - & - & + & - & \cdots & - \\ - & - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & - & \cdots & + \end{array} \right|$$

$$= (+)\mathbf{e}_0 + \sum_{i=1}^n (-1)^i d_i^n \mathbf{e}_i$$

and

$$d_1^n = (-(+)) - (-d_1^{n-1}) + (-d_2^{n-2}) - \cdots + (-1)^{n-1} (-d_{n-1}^{n-1})$$

$$= (-) + \sum_{j=1}^{n-1} (-1)^{j+1} (d_j^{n-1}).$$

We now proceed by induction. We first consider the case where $k = 1$. We can see that

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 \\ - & + \end{vmatrix} = (+)\mathbf{e}_0 - (-)\mathbf{e}_1 = (+)\mathbf{e}_0 + (+)\mathbf{e}_1,$$

so the lemma holds for $k = 1$. Now suppose that the lemma holds for $k - 1$, and consider V^k . The \mathbf{e}_0 coordinate is positive, by Lemma 2.3 on page 5. The \mathbf{e}_i coordinate is given by

$$\begin{aligned} (-1)^i d_i^k &= (-1)^i (-1)^{i-1} d_1^k = (-1) \left((-1) + \sum_{j=1}^{k-1} (-1)^{j+1} (d_j^{k-1}) \right) \\ &= (+) + \underbrace{\sum_{j=1}^{k-1} (-1)^j (d_j^{k-1})}_{\mathbf{e}_j \text{ term of } V^{k-1}} = (+), \end{aligned}$$

so the lemma holds for k , and so holds for all $k \in \mathbb{N}$. \square

Corollary 2.5. *For each $p \in \Pi$, the kernel of*

$$M(p) = \begin{pmatrix} Q_1(p_0) & Q_1(p_1) & Q_1(p_2) & \cdots & Q_1(p_n) \\ Q_2(p_0) & Q_2(p_1) & Q_2(p_2) & \cdots & Q_2(p_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_n(p_0) & Q_n(p_1) & Q_n(p_2) & \cdots & Q_n(p_n) \end{pmatrix}$$

is one dimensional and spanned by a vector with strictly positive entries. Further, we can define a continuous function $\kappa : \Pi \rightarrow \mathbb{R}^{n+1}$ such that $\kappa(p)$ is entry-wise positive, and $\kappa(p)$ is in the kernel of $M(p)$.

Proof. We can see that $M(p)$ is always rank n , as the right hand $n \times n$ sub-matrix is invertible, by Lemma 2.3, so its kernel is everywhere rank one. If at each $p \in \Pi$ we take the V^n defined earlier, and define this as $\kappa(p)$, it is clear that this is entry-wise positive, orthogonal to the span of the row vectors (so in the kernel of the operator), and has entries that sum to one, from the definitions and the above proved theorems. \square

2.3. Canonical Analytic Functions. For $p \in \Pi$ we define

$$k_p = \sum_{j=0}^n \kappa_j(p) \mathbb{P}(\cdot, p_j),$$

where κ is as in corollary 2.5. Define $\tau : \Pi \rightarrow \mathbb{R}^{n+1}$ by $\tau(p) = \kappa(p)/k_p(b)$. We then define

$$h_p = \sum_{j=0}^n \tau_j(p) \mathbb{P}(\cdot, p_j).$$

It is clear that this corresponds to the measure

$$\mu = \sum_{j=0}^n \tau_j(p) \delta_{p_j}$$

on B . We can see that h_p , thus defined, is a positive harmonic function, with $h_p(b) = 1$. We can also see that its periods are zero, as

$$(2.1) \quad P_j(h_p) = \int_B Q_j d\mu = \int_B Q_j \sum_{i=0}^n \tau_i(p) \delta_{p_i} = \sum_{i=0}^n \tau_i(p) \int_B Q_j \delta_{p_i} = \sum \tau_i(p) Q_j(p_i) = 0,$$

as $\tau(p)$ is in the kernel of $M(p)$, and (2.1) is just the j -th coordinate of $M(p)\tau(p)$. The function h_p is therefore the real part of an analytic function f_p on R . We require that $f_p(b) = 1$.

We define $\mathcal{H}(R)$ as the space of holomorphic functions on R , with the compact open topology. This is locally convex, metrisable, and has the Heine-Borel property, that is, closed bounded subsets of $\mathcal{H}(R)$ are compact. We then define

$$\mathbb{K} = \{f \in \mathcal{H}(R) : f(b) = 1, f + \bar{f} > 0\}.$$

Lemma 2.6. *The set \mathbb{K} is compact.*

Proof. \mathbb{K} is clearly closed, so it suffices to show that \mathbb{K} is bounded. The case where R is the unit disc is proved in [DM05], and we use this result without proof.

Since the B_0, \dots, B_n are disjoint, closed sets, and R is T_4 , we can find disjoint open sets U_0, \dots, U_n containing each. By a simple topological argument we can show that there exists some $E > 0$ such that

$$O_i(E) := \{z \in \mathbb{C} : d(z, B_i) < E\} \subseteq U_i.$$

It is clear that R is covered by the family of connected compact sets

$$\{K_\epsilon\} := \left\{ R \setminus \left(\bigcup_i O_i(\epsilon) \right) : 0 < \epsilon < E \right\},$$

so it is sufficient to work with just these compact sets.

We choose a sequence of disjoint, simple paths v_0, \dots, v_n through X such that v_i goes from B_i to B_{i+1} , and v_0 passes through b (note that when X is a symmetric domain, $v_i = \mathbb{X}_i$ satisfies this). It is clear that the union of these paths cuts X into two disjoint, simply connected sets U and V . It is also possible to show that we can choose a $\delta > 0$ such that adding

$$W := \{z \in R : d(z, v_i) \leq \delta \text{ for some } i\}$$

to either of these sets preserves simple connectivity. We can see that $K_\epsilon^+ := K_\epsilon \cap (U \cup W)$ and $K_\epsilon^- := K_\epsilon \cap (V \cup W)$ are simply connected compact sets containing b , whose union is K_ϵ . By the Riemann mapping theorem, we can canonically map K_ϵ^\pm to the unit disc, in a way that takes b to zero, so by the result of [DM05] mentioned earlier, we have a constant M_ϵ^\pm , such that f analytic on R with $f(b) = 1$ implies for all $z \in K_\epsilon^\pm$, $|f(z)| \leq M_\epsilon^\pm$. \square

Lemma 2.7. *The extreme points of \mathbb{K} are precisely $\{f_p : p \in \Pi\}$.*

Proof. Clearly, each f_p is an extreme point of \mathbb{K} , so we prove the converse – if $f \neq f_p$, then f is not an extreme point of \mathbb{K} .

If $f \in \mathbb{K}$, then the real part of f is a positive harmonic function h with $h(b) = 1$. We therefore know that there is some positive measure μ on B such that

$$h(w) = \int_B \mathbb{P}(w, z) d\mu(z).$$

As f is holomorphic, by Corollary 2.2 on page 5, μ must support at least one point on each B_i . If $f \neq f_p$, then μ must support more than one point on some B_i .

Now, a note. We know f is holomorphic if $P_j(h) = 0$ for $j = 0, \dots, n$. However, we know that $\sum_{j=0}^n Q_j = 0$, so $\sum_{j=0}^n P_j(h) = 0$, so if we show that all but one of the $P_j(h)$ are zero, we have shown that they are all zero, so f is holomorphic.

With that in mind, suppose that μ supports more than one point on B_0 . We do not lose any generality by doing this, as relabelling the boundary curves does not matter in the proof below, so we can safely relabel any given boundary curve B_0 . We divide B_0 into two parts, A_1 and A_2 , in such a way that μ is non-zero on both.

Now, let

$$a_{jl} = \int_{A_l} Q_j d\mu, \quad l = 1, 2,$$

and

$$k_{jm} = \int_{B_m} Q_j d\mu, \quad m = 1, \dots, n,$$

Since h is the real part of an analytic function,

$$0 = \int_B Q_j d\mu,$$

so

$$\sum_{m=1}^n k_{jm} + a_{j1} + a_{j2} = 0.$$

Since $Q_j < 0$ on B_i for $i \neq j$, for any $M \subseteq \{1, \dots, n\}$ containing j ,

$$\sum_{m \in M} k_{jm} = - \left(a_{j1} + a_{j2} + \sum_{m \notin M} k_{jm} \right) > 0.$$

By cyclically permuting the $m - j$ rows in the middle we get

$$(-1)^{j-m-1}(-1)^{j+m} \left(\begin{array}{cccc|c|cccc} & & & & \text{col } j & & & & & \\ & + & - & \cdots & - & & & & & \\ & - & + & \cdots & - & & & & & \\ & \vdots & \vdots & \ddots & \vdots & - & & & - & \\ & - & - & \cdots & + & & & & & \\ \hline \text{row } j & & & & - & - & & & - & \\ \hline & & & & & & + & - & \cdots & - \\ & & & & & & - & + & \cdots & - \\ & & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & & - & - & \cdots & + \end{array} \right),$$

and by cyclically permuting the first j rows, and the first j columns we get

$$(-1)(-1)^{j-1}(-1)^{j-1}d_1^{n-1},$$

which we note is precisely the e_1 term of V^{n-1} in Lemma 2.4 on page 6, which is positive.

If $j > m$, then c_{jm} is given by

$$(2.3) \quad (-1)^{j+m} \left(\begin{array}{cccc|c|cccc} + & - & \cdots & - & & & & & & \\ - & + & \cdots & - & & & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & & & \\ - & - & \cdots & + & & & & & & \\ \hline & & & & \underbrace{\hspace{10em}}_{(j-m) \times (j-m) \text{ block}} & & & & & \\ & & & & - & - & \cdots & - & & \\ & & & & + & - & - & \ddots & \vdots & \\ & & & & - & + & - & \ddots & - & \\ & & & & \vdots & \vdots & \ddots & \ddots & - & \\ & & & & - & \cdots & - & + & - & \\ \hline & & & & & & & & + & - & \cdots & - \\ & & & & & & & & - & + & \cdots & - \\ & & & & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & & - & - & \cdots & + \end{array} \right).$$

But note that transposing matrices preserves determinant, and the transpose of the matrix in (2.3) is the matrix in (2.2), so $c_{jm} = c_{mj}$, which we already know is positive. Therefore, K^{-1} has all positive entries. Since

$$\begin{pmatrix} -a_{1l} \\ \vdots \\ -a_{nl} \end{pmatrix}$$

has all positive entries, we define

$$\begin{pmatrix} b_{1l} \\ \vdots \\ b_{nl} \end{pmatrix} := K^{-1} \begin{pmatrix} -a_{1l} \\ \vdots \\ -a_{nl} \end{pmatrix}.$$

Define positive measures ν_1, ν_2 by

$$\nu_l(A) = \mu(A \cap A_l) + \sum_{m=1}^n b_{ml} \mu(A \cap B_m).$$

Then

$$\int_B Q_j d\nu_l = a_{jl} + \sum_{m=1}^n k_{jm} b_{ml} = 0,$$

so each

$$h_l = \int_B \mathbb{P}(\cdot, w) d\nu_l(w), \quad l = 1, 2,$$

is the real part of an analytic function g_l with $\Im g_l(b) = 0$. We can see that $\nu_1 + \nu_2 = \mu$ as

$$K \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{m=1}^n \int_{B_m} Q_1 d\mu \\ \vdots \\ \sum_{m=1}^n \int_{B_m} Q_n d\mu \end{pmatrix} = \begin{pmatrix} P_1(\mathfrak{h}) - a_{11} - a_{12} \\ \vdots \\ P_n(\mathfrak{h}) - a_{n1} - a_{n2} \end{pmatrix}.$$

Multiplying both sides by K^{-1} gives $b_{m1} + b_{m2} = 1$. We therefore have $h_1 + h_2 = h$. Thus, $g_l/g_l(b) \in \mathbb{K}$ and

$$f = g_1(b) \left(\frac{g_1}{g_1(b)} \right) + g_2(b) \left(\frac{g_2}{g_2(b)} \right),$$

so f is a convex combination of two other points in \mathbb{K} . Hence, f is not an extreme point. \square

Lemma 2.8. *The set $\widehat{\mathbb{K}}$ of extreme points of \mathbb{K} is a closed set, and the function taking Π to \mathbb{K} by $p \mapsto f_p$ is a homeomorphism onto $\widehat{\mathbb{K}}$.*

Proof. The proof is exactly as that of Lemma 2.11 in [DM05]. \square

2.4. Test Functions. For $p \in \Pi$, define

$$\psi_p = \frac{f_p - 1}{f_p + 1}.$$

The real part, h_p , of f_p is harmonic across $B \setminus \{p_0, \dots, p_n\}$, therefore f_p is analytic across $B \setminus \{p_0, \dots, p_n\}$. Also, f_p looks locally like $g_j/(z - p_j)$ at p_j , for some analytic g_j , non-vanishing at p_j (by [Fis83, Ch. 4, Prop. 6.4]). We can see from this that ψ_p is continuous onto B and $|\psi_p| = 1$ on B .

By the reflection principle, ψ_p is inner and extends analytically across B , and $\psi_p^{-1}\{1\} = \{p_0, \dots, p_n\}$, so the preimage of each point $z \in \mathbb{D}$ is exactly $n + 1$ points, up to multiplicity, and so ψ_p has $n + 1$ zeroes.

Similarly, if ψ is analytic in a neighbourhood of R , with modulus one on B and $n + 1$ zeroes in R , then $\psi^{-1}\{1\}$ has $n + 1$ points. Also, the real part of

$$f = \frac{1 + \psi}{1 - \psi}$$

is a positive harmonic function which is zero on B except where $\psi(z) = 1$. By Corollary 2.2 on page 5, f cannot be identically zero on any B_i , so there must be one point from $\psi^{-1}\{1\}$ on each B_i . If, further, $\psi(b) = 0$, then $\psi = \psi_p$ for some $p \in \Pi$.

We define $\Theta = \{\psi_p : p \in \Pi\}$.

Theorem 2.9. *If ρ is analytic in R and if $|\rho| \leq 1$ on R , then there exists a positive measure μ on Π and a measurable function h defined on Π whose values are functions $h(\cdot, p)$ analytic in R so that*

$$1 - \rho(z)\overline{\rho(w)} = \int_{\Pi} h(z, p) [1 - \psi_p(z)\overline{\psi_p(w)}] \overline{h(w, p)} d\mu(p).$$

Proof. First suppose $\rho(b) = 0$.

Let

$$f = \frac{1 + \rho}{1 - \rho}$$

so

$$\rho = \frac{f - 1}{f + 1}.$$

Hence

$$(2.4) \quad 1 - \rho(z)\overline{\rho(w)} = 2 \frac{f(z) + \overline{f(w)}}{(f(z) + 1)(\overline{f(w)} + 1)}$$

Since h , the real part of f , is positive and $f(b) = 1$, the function f is in \mathbb{K} . Since \mathbb{K} is a compact convex subset of the locally convex topological vector space $\mathcal{H}(R)$, by the Krein-Milman theorem, f is in the closed convex hull of $\widehat{\mathbb{K}} = \{f_p : p \in \Pi\}$, the set of extreme points of \mathbb{K} . Therefore, there exists some regular Borel probability measure ν on Π such that

$$f = \int_{\Pi} f_p d\nu(p).$$

Using the definition of ψ_p and (2.4), we can show that

$$1 - \rho(z)\overline{\rho(w)} = \int_{\Pi} \frac{1 - \psi_p(z)\overline{\psi_p(w)}}{(f(z) + 1)(1 - \psi_p(z))(1 - \overline{\psi_p(w)})(\overline{f(w)} + 1)} d\nu(p).$$

Finally, if $\rho(b) = a$, then we have a representation like the one above, as

$$1 - \left(\frac{\rho(z) - a}{1 - \bar{a}\rho(z)} \right) \overline{\left(\frac{\rho(w) - a}{1 - \bar{a}\rho(w)} \right)} = \frac{(1 - a\bar{a})(1 - \rho(z)\overline{\rho(w)})}{(1 - \bar{a}\rho(z))(1 - a\rho(w))}.$$

□

The interested reader may note that the set Θ is a collection of test functions for $H^\infty(R)$, as defined in [DM07].

Note 2.10. We have used $n+1$ parameters to describe the inner functions in Θ , however, we only need n , as we can identify them with the inner functions with $n+1$ zeroes, by the argument in the introduction to Section 2.4 on page 13. If we then fix some $\tilde{p}_0 \in B_0$, it is then clear that for all $p \in \Pi$, $\overline{\psi_p(\tilde{p}_0)\psi_p}$ is an inner function with $n+1$ zeroes, with one of them at b , and $\overline{\psi_p(\tilde{p}_0)\psi_p(\tilde{p}_0)} = 1$, so $\overline{\psi_p(\tilde{p}_0)\psi_p} = \psi_q$, where $q = (\tilde{p}_0, q_1, \dots, q_n)$, for some $q_1 \in B_1, \dots, q_n \in B_n$. We define

$$\tilde{\Theta} := \left\{ \psi_q : q = (\tilde{p}_0, q_1, \dots, q_n), q_1 \in B_1, \dots, q_n \in B_n \right\},$$

which is also a set of test functions for $H^\infty(R)$.

3. MATRIX INNER FUNCTIONS

3.1. Preliminaries.

Theorem 3.1. *If R is symmetric, then there is some $b \in \mathbb{X}$, and some $\psi_p \in \tilde{\Theta}$ with $n+1$ distinct zeroes b, z_1, \dots, z_n , where $z_1, \dots, z_n \notin \mathbb{X}$, and $z_i \neq \omega(z_j)$ for all i, j .*

Proof. For now, choose a $b_0 \in R$, and use this as our b . We will find a better choice for b later in the proof. Take p_0^- as \tilde{p}_0 , and use this to define $\tilde{\Theta}$ as in Note 2.10. We will give this $\tilde{\Theta}$ an unusual name, $\tilde{\Theta}_0$, and call the functions in it φ_p , rather than ψ_p . This is to distinguish it from the $\tilde{\Theta}$ and ψ_p in the statement of the theorem, which we will construct later.

Choose some $p_1 \in B_1 \setminus \mathbb{X}, \dots, p_n \in B_n \setminus \mathbb{X}$. Consider the path v along \mathbb{X} from B_1 to B_0 . Its image under φ_p is a path leading to 1. We can see that $\varphi_p^{-1}\{1\}$ has $n+1$ points. As X is Hausdorff and locally connected, there are disjoint, connected open sets U_0, U_1, \dots, U_n around each of these points, and since φ_p is an open mapping on each of these open sets,

$$\mathcal{N} := \bigcap_{i=0}^n \varphi_p(U_i)$$

is a (relatively) open neighbourhood of 1, whose preimage is $n+1$ disjoint open sets, U'_0, \dots, U'_n . Also, we can choose U_1, \dots, U_n such that none of them intersects \mathbb{X} , and none of them intersects any $\omega(U_i)$ (since $p_1, \dots, p_n \notin \mathbb{X}$, and \mathbb{X} closed). Now, we can lift $\varphi_p(v) \cap \mathcal{N}$ to each of these U'_i , we choose a point $y \in \varphi_p(v) \cap \mathcal{N}$, and note that $\varphi_p^{-1}\{y\}$ has exactly $n+1$ distinct points, none of which maps to another under ω , and exactly one of which is on \mathbb{X} . The point on \mathbb{X} , we use as our b for the rest of the proof. We take a Möbius transform m which preserves the unit circle, and maps y to 0, and notice that $m \circ \varphi_p$ is an inner function which has $n+1$ zeroes, exactly one of which, b , is on \mathbb{X} . If we define $\tilde{\Theta}$ using our new b , and $\tilde{p}_0 = p_0^-$, then $\overline{m \circ \varphi_p(p_0^-)} m \circ \varphi_p \in \tilde{\Theta}$, and has the required zeroes, and so is our ψ_p . \square

Remark 3.2. Note that in the above argument, we can choose our b as close to p_0^- as we like, so in particular, we can choose b such that $h_0(b) > 1/2$. By an argument similar to that in [DM05, Prop. 2.13], we can see that no $\psi_p \in \widetilde{\Theta}$ has all its zeroes at b .

Theorem 3.3. *If R is symmetric, then $Q_j(p_i) = \eta(p_i) Q_j(\omega(p_i))$, for some $\eta : B \rightarrow \mathbb{C}$ which does not depend on j .*

Proof. We write Q_j as

$$Q_j(p) = \frac{\partial h_j}{\partial n_p}(p)$$

where $\partial/\partial n_p$ is the normal derivative at p . We also define $\partial/\partial t_p$ as the tangent derivative at p .

Now, note that if h is harmonic and ω is anticonformal, then $h \circ \omega$ is also harmonic, and since h_j and $h_j \circ \omega$ have the same values on B , they must be equal, so

$$\frac{\partial h_j(p_i)}{\partial n_{p_i}} = \frac{\partial h_j(\omega(p_i))}{\partial n_{p_i}},$$

and so

$$\begin{aligned} Q_j(p_i) &= \frac{\partial h_j(p_i)}{\partial n_{p_i}} \\ &= \frac{\partial h_j(\omega(p_i))}{\partial n_{\omega(p_i)}} \cdot \frac{\partial n_{\omega(p_i)}}{\partial n_{p_i}} + \frac{\partial h_j(\omega(p_i))}{\partial t_{\omega(p_i)}} \cdot \frac{\partial t_{\omega(p_i)}}{\partial n_{p_i}} \\ &= Q_j(\omega(p_i)) \cdot \underbrace{\frac{\partial n_{\omega(p_i)}}{\partial n_{p_i}}}_{\eta(p_i)}. \end{aligned}$$

□

Lemma 3.4. *If η is defined as above, and $b \in \mathbb{X}$ then*

$$\mathbb{P}(b, p_j) = \eta(p_j) \mathbb{P}(b, \omega(p_j)).$$

Proof. We can write

$$\mathbb{P}(b, p_j) = \frac{d\omega_b(p_j)}{ds(p_j)} \quad \text{and} \quad \mathbb{P}(b, \omega(p_j)) = \frac{d\omega_b(\omega(p_j))}{ds(\omega(p_j))},$$

and note that if h is harmonic, then $h \circ \omega$ is harmonic, and $h \circ \omega(b) = h(b)$. So, for any measurable set $E \subseteq B$,

$$\omega_b(E) = \omega_b(\omega(E)),$$

so $d\omega_b(p_j) = d\omega_b(\omega(p_j))$. Hence,

$$\begin{aligned} \mathbb{P}(b, p_j) &= \frac{d\omega_b(p_j)}{ds(p_j)} = \frac{d\omega_b(\omega(p_j))}{ds(p_j)} = \frac{ds(\omega(p_j))}{ds(p_j)} \cdot \frac{d\omega_b(\omega(p_j))}{ds(\omega(p_j))} \\ &= \frac{dn_{\omega(p_j)}}{dn_{p_j}} \cdot \mathbb{P}(b, \omega(p_j)) = \eta(p_j) \mathbb{P}(b, \omega(p_j)), \end{aligned}$$

since

$$\frac{ds(\omega(p_j))}{ds(p_j)} \underbrace{=}_{\star} - \frac{dt_{\omega(p_j)}}{dt_{p_j}} \underbrace{=}_{\dagger} \frac{dn_{\omega(p_j)}}{dn_{p_j}},$$

where \star is due to the fact that ω is sense reversing, and \dagger is due to the Cauchy-Riemann equation for anti-holomorphic maps. \square

Definition 3.5. We say a holomorphic 2×2 matrix valued function F on R has a *standard zero set* if

- (1) F has distinct zeroes b, a_1, \dots, a_{2n} , where $F(b) = 0$, and $\det(F)$ has zeroes of multiplicity one at each of a_1, \dots, a_{2n} ;
- (2) if $\gamma_j \neq 0$ are such that $F(a_j)^* \gamma_j = 0$, $j = 1, \dots, 2n$, then no $n+1$ of the γ_j lie on the same complex line through the origin;
- (3) $Ja_j \neq P_i$ for $j = 1, \dots, 2n$, $i = 1, \dots, n$, where P_1, \dots, P_n are the poles of the Fay kernel $K^b(\cdot, z)$.

We have not defined K^b yet, and will not do so until Section 4. For now, all we need to know about K^b is that all its poles are on $J(\mathbb{X})$.

3.2. The construction. We take $\psi_{\mathbf{p}}$ as in Theorem 3.1 on page 15. Note that $\psi_{\mathbf{p}} \circ \omega$ is an inner function with zeroes at $b, \omega(z_1), \dots, \omega(z_n)$, equal to one at $p_0^-, \omega(\mathbf{p}_1), \omega(\mathbf{p}_2), \dots, \omega(\mathbf{p}_n)$, so must equal $\psi_{\omega(\mathbf{p})}$.

Definition 3.6. We say S is a *team of projections* if S is a collection of n pairs of non-zero orthogonal projections on \mathbb{C}^2 , (P^{j+}, P^{j-}) , such that

$$P^{1+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^{1-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{j+} + P^{j-} = I, \quad j = 1, \dots, n.$$

Let S_0 be the *trivial team*, given by $P^{j\pm} = P^{1\pm}$ for all j .

We define

$$H_{S,p} = \tau_0(p) \mathbb{P}(\cdot, p_0^-) I + \sum_{i=1}^n \tau_i(p) \left[\mathbb{P}(\cdot, p_i) P^{i+} + \eta(p_i) \mathbb{P}(\cdot, \omega(p_i)) P^{i-} \right].$$

We note that, by Lemma 3.4,

$$\begin{aligned} H_{S,p}(b) &= \tau_0(p) \mathbb{P}(b, p_0^-) I + \sum_{i=1}^n \tau_i(p) [\mathbb{P}(b, p_i) I] \\ &= \left[\sum_{i=0}^n \tau_i(p) \mathbb{P}(b, p_i) \right] I = h_{\mathbf{p}}(b) I = I. \end{aligned}$$

For $x \in \mathbb{C}^2$ a unit vector, $\langle H_{S,p} x, x \rangle$ corresponds to the measure

$$\mu_{x,x} = \tau_0 \delta_{p_0^-} + \sum_{i=1}^n \tau_i \cdot \left[\delta_{p_i} \|P^{i+} x\| + \delta_{\omega(p_i)} \eta(p_i) \|P^{i-} x\| \right],$$

so

$$\begin{aligned} \int_B Q_j d\mu_{x,x} &= \tau_0 Q_j(p_0^-) + \sum_{i=1}^n \tau_i \left[Q_j(p_i) \|P^{i+} x\| + \eta(p_i) Q_j(\omega(p_i)) \|P^{i-} x\| \right] \\ &= \tau_0 Q_j(p_0^-) + \sum_{i=1}^n \tau_i Q_j(p_i) \|x\| \\ &= 0, \end{aligned}$$

by definition of τ .

Hence, $\langle H_{S,p} x, x \rangle$ is the real part of an analytic function, so $H_{S,p}$ is the real part of a holomorphic 2×2 matrix function $G_{S,p}$, normalised by $G_{S,p}(b) = I$.

We now define

$$\Psi_{S,p} = (G_{S,p} - I) \cdot (G_{S,p} + I)^{-1}.$$

Lemma 3.7. *If \mathbf{p} is as in Theorem 3.1 on page 15, for each S :*

- (1) $\Psi_{S,\mathbf{p}}$ is analytic in a neighbourhood of X and unitary valued on B ;
- (2) $\Psi_{S,\mathbf{p}}(b) = 0$;
- (3) $\Psi_{S,\mathbf{p}}(p_0^-) = I$;
- (4) $\Psi_{S,\mathbf{p}}(\mathbf{p}_1)e_1 = e_1$ and $\Psi_{S,\mathbf{p}}(\omega(\mathbf{p}_1))e_2 = e_2$;
- (5) $\Psi_{S,\mathbf{p}}(\mathbf{p}_i)P^{i+} = P^{i+}$ and $\Psi_{S,\mathbf{p}}(\omega(\mathbf{p}_i))P^{i-} = P^{i-}$;
- (6) $\Psi_{S_0,\mathbf{p}} = \begin{pmatrix} \psi_{\mathbf{p}} & 0 \\ 0 & \psi_{\omega(\mathbf{p})} \end{pmatrix}$.

Proof. Thinking about $\mathbb{P}(z, r)$ as a function of z , in a neighbourhood of $r \in B$, the Poisson kernel $\mathbb{P}(z, r)$ is the real part of some function of the form $g_r(z)(z-r)^{-1}$, where g_r is analytic in the neighbourhood, and non-vanishing at r (by [Fis83, Ch. 4, Prop. 6.4]). At any other point $q \in B$, $\mathbb{P}(z, r)$ extends to a harmonic function on a neighbourhood of q , so must be the real part of some analytic function, with real part 0 at q .

We can see that if $r \in B$ is not p_0^- , $\mathbf{p}_1, \dots, \mathbf{p}_n, \omega(\mathbf{p}_1), \dots, \omega(\mathbf{p}_n)$, then $G_{S,\mathbf{p}}$ is analytic in a neighbourhood of r . Further, $G_{S,\mathbf{p}} + I$ is invertible near r as $G_{S,\mathbf{p}}(z) = H_{S,\mathbf{p}}(z) + iA(z)$ for some self-adjoint matrix valued function $A(z)$, and $H_{S,\mathbf{p}}(r) = 0$. Thus, $G_{S,\mathbf{p}} + I$ is invertible at and, by continuity, near r . We have

$$I - \Psi_{S,\mathbf{p}} \Psi_{S,\mathbf{p}}^* = 2(G_{S,\mathbf{p}} + I)^{-1} \underbrace{(G_{S,\mathbf{p}} + G_{S,\mathbf{p}}^*)}_{iA + (iA)^* = 0} (G_{S,\mathbf{p}} + I)^{*-1},$$

which is zero at r , so $\Psi_{S,\mathbf{p}}$ must be unitary at r .

From the definition of $G_{S,\mathbf{p}}$, in a neighbourhood of p_0^- , there are analytic functions g_1, g_2, k_1, k_2 so that the real parts of k_j are 0 at p_0^- , each g_j is

non-vanishing at p_0^- , and

$$G_{S,\mathbf{p}}(z) = \begin{pmatrix} \frac{g_1(z)}{z-p_0^-} & k_1(z) \\ k_2(z) & \frac{g_2(z)}{z-p_0^-} \end{pmatrix},$$

so

$$\begin{aligned} (G_{S,\mathbf{p}}(z) + I)^{-1} &= \frac{1}{\frac{g_1+z-p_0^-}{z-p_0^-} \frac{g_2+z-p_0^-}{z-p_0^-} - k_1(z)k_2(z)} \begin{pmatrix} \frac{g_2(z)-z-p_0^-}{z-p_0^-} & -k_1(z) \\ -k_2(z) & \frac{g_1(z)-z-p_0^-}{z-p_0^-} \end{pmatrix} \\ &= \frac{\begin{pmatrix} (g_2(z) - z - p_0^-)(z - p_0^-) & -k_1(z)(z - p_0^-)^2 \\ -k_2(z)(z - p_0^-)^2 & (g_1(z) - z - p_0^-)(z - p_0^-) \end{pmatrix}}{(g_1(z) - z - p_0^-)(g_2(z) - z - p_0^-) - k_1(z)k_2(z)(z - p_0^-)^2}. \end{aligned}$$

Note that the denominator is non-zero at and near p_0^- , so $G_{S,\mathbf{p}} + I$ is invertible. We can use this to calculate $\Psi_{S,\mathbf{p}}$ directly¹, and show that $\Psi_{S,\mathbf{p}}$ is analytic in a neighbourhood of p_0^- , and $\Psi_{S,\mathbf{p}}(p_0^-) = I$, so we have (3).

Now we look at \mathbf{p}_1 . Near \mathbf{p}_1 we have analytic functions g, k_1, k_2, k_3 , on a neighbourhood of \mathbf{p}_1 , where k_1, k_2, k_3 have zero real part at \mathbf{p}_1 , g is non-zero at \mathbf{p}_1 , and

$$G_{S,\mathbf{p}}(z) = \begin{pmatrix} \frac{g(z)}{z-\mathbf{p}_1} & k_1 \\ k_2 & k_3 \end{pmatrix}.$$

Since $k_3 + 1$ has real part 1 at \mathbf{p}_1 , $g(z)(z - \mathbf{p}_1)^{-1}$ has a pole, and k_1, k_2 are analytic at \mathbf{p}_1 , we see that $G_{S,\mathbf{p}} + I$ is invertible near \mathbf{p}_1 . By direct computation, we see that $\Psi_{S,\mathbf{p}}$ is analytic in a neighbourhood of \mathbf{p}_1 and

$$\Psi_{S,\mathbf{p}}(\mathbf{p}_1) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k_3(\mathbf{p}_1)-1}{k_3(\mathbf{p}_1)+1} \end{pmatrix}.$$

A similar argument holds for $\omega(\mathbf{p}_1)$, so we have (4), and by working in the orthonormal basis induced by P^{j+} and P^{j-} , (5) follows. Also, we have now shown $\Psi_{S,\mathbf{p}}$ is analytic at every point, so (1) follows.

(6) and (2) follow easily from the definitions. \square

Lemma 3.8. *We define $\|S_1 - S_2\|_\infty = \max_{j\pm} \|P_1^{j\pm} - P_2^{j\pm}\|$, giving a metric on the space \mathcal{T} of all teams of projections. There exists some non-trivial sequence $S_m \rightarrow S_0$ such that for all m , $\Psi_{S_m,\mathbf{p}}$ has a standard zero set.*

Proof. Since the zeroes of $\psi_{\mathbf{p}}$ and $\psi_{\omega(\mathbf{p})}$ are all distinct except for b , it is clear that

$$\Psi_{S_0,\mathbf{p}} = \begin{pmatrix} \psi_{\mathbf{p}} & 0 \\ 0 & \psi_{\omega(\mathbf{p})} \end{pmatrix}$$

has a standard zero set.

¹The calculation is omitted, but can be readily verified by hand, or with a computer algebra system

We note that whatever value we take for ϵ , there is an $S \neq S_0$ within ϵ of S_0 , so there is some non-trivial sequence S_m converging to S_0 .

The sequence $\Psi_{S_m, \mathbf{p}}$ is uniformly bounded, so has a sub-sequence Ψ_m which converges uniformly on compact subsets of R to some Ψ . This means

$$G_m = (I + \Psi_m)(I - \Psi_m)^{-1}$$

converges uniformly on compact subsets of R to

$$G = (I + \Psi)(I - \Psi)^{-1}.$$

H_m , the real part of G_m is harmonic, and

$$H_m - H_0 = \sum_{i=2}^n \tau_i(\mathbf{p}) \mathbb{P}(\cdot, \mathbf{p}_i) \left[P_m^{i+} - P^{1+} \right] + \tau_i(\bar{\omega}(\mathbf{p})) \mathbb{P}(\cdot, \bar{\omega}(\mathbf{p}_i)) \left[P_m^{i-} - P^{1-} \right].$$

Since $P_m^{i\pm} \rightarrow P^{1\pm}$, we see that $H_m \rightarrow H_0$, and since $G(b) = I = G_0(b)$, $G_m \rightarrow G_0$, so $\Psi = \Psi_0$, and $\Psi_m \rightarrow \Psi_0$ uniformly on compact sets.

Let $d_m(z) = \det(\Psi_m(z))$. This is analytic, and unimodular on B . Draw small, disjoint circles in R around the zeroes of d_0 (which correspond to the zeroes of Ψ_0). By Hurwitz's theorem, there exists some M such that for all $m \geq M$, d_m and d_0 have the same number of zeroes in each of these circles, so the zeroes of d_m must be distinct, apart from the repeated zero at b . In particular, the zeroes $(b, a_1^m, \dots, a_{2n}^m)$ of Ψ_m converge to the zeroes $(b, a_1^0, \dots, a_{2n}^0)$ of Ψ_0 .

Finally, if $\|\gamma_1^m\| = 1$, $\Psi_m(a_1^m)^* \gamma_1^m = 0$ and a_1^m is close to a_1^0 , then

$$\Psi_0(a_1^0)^* \gamma_1^m = \left(\Psi_0(a_1^0) - \Psi_0(a_1^m) \right)^* \gamma_1^m + \left(\Psi_0(a_1^m) - \Psi_m(a_1^m) \right)^* \gamma_1^m.$$

However, the right hand side tends to zero as m tends to infinity, so $\Psi_0(a_1^0)^* \gamma_1^m$ tends to zero. Since γ_1^m is a bounded sequence in a finite-dimensional complex space, it has a convergent sub-sequence, which we shall also call γ_1^m . This γ_1^m must converge to something in the kernel of $\Psi_0(a_1^0)^*$, that is, a multiple of e_1 . We apply this argument to a_2, \dots, a_{2n} , and find a sub-sequence such that n of the γ_i^m s tend to multiples of e_1 and n of them tend to multiples of e_2 , so for m big enough, no $n+1$ of them are collinear. \square

4. THETA FUNCTIONS

4.1. The Jacobian Variety. We know that for each $i = 1, \dots, n$, h_i is locally the real part of an analytic function g_i . The differential dg_i can be extended from R to Y (as in Theorem 1.1, Y is the Schottky double of R), and

$$\alpha_i := \frac{1}{2} dg_i, \quad i = 1, \dots, n$$

is then a basis for the space of holomorphic 1-forms on Y . We see that if we define a homology basis for Y by $A_j = \mathbb{X}_j - J(\mathbb{X}_j)$ and B_j as before, then

$\int_{A_j} \alpha_i = \delta_{ij}$ and

$$\Omega := \left(\int_{B_j} \alpha_i \right)_{ij}$$

has positive definite imaginary part (see, for example, [FK92, III.2.8]).

We define a lattice

$$L := \mathbb{Z}^n + \Omega \mathbb{Z}^n \subseteq \mathbb{C}^n$$

define the *Jacobian variety* by

$$\mathcal{J}(Y) := \mathbb{C}^n / L,$$

and define the *Abel-Jacobi maps* $\chi : Y \rightarrow \mathbb{C}^n$ and $\chi_0 : Y \rightarrow \mathcal{J}(Y)$ by

$$\chi(y) := \begin{pmatrix} \int_{p_0^-}^y \alpha_1 \\ \vdots \\ \int_{p_0^-}^y \alpha_n \end{pmatrix}, \quad \chi_0(y) = [\chi(y)].$$

Note that the integral depends on the path integrated over. However, any two paths differ only by a closed path, and $A_1, \dots, A_n, B_1, \dots, B_n$ is a homology basis for Y , so any closed path is homologous to a sum of paths in this basis. Also,

$$\int_{A_j} \alpha_i, \int_{B_j} \alpha_i \in L, \text{ so } \left[\int_{A_j} \alpha_i \right] = \left[\int_{B_j} \alpha_i \right] = 0,$$

so the choice of path to integrate over does not affect $\chi_0(y)$.

Proposition 4.1. *The Abel-Jacobi map has the following properties:*

- (1) χ_0 is a one-one conformal map of Y onto its image in $\mathcal{J}(Y)$; and
- (2) $\chi_0(Jy) = -\chi_0(y)^*$, where $*$ denotes the coordinate-wise conjugate.

Proof. (1) is proved in [FK92, III.6.1], (2) holds because $p_0^- \in \mathbb{X}$ and

$$g_j(Jy) - g_j(p_0^-) = -\overline{(g_j(y) - g_j(p_0^-))}.$$

□

4.2. Theta Functions.

Definition 4.2. Roughly following [Mum83], we define the *theta function* $\vartheta : \mathbb{C}^n \rightarrow \mathbb{C}$ by

$$\vartheta(z) = \sum_{m \in \mathbb{Z}^n} \exp(\pi i \langle \Omega m, m \rangle + 2\pi i \langle z, m \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the usual \mathbb{C}^n inner product. This function is quasi-periodic, as

$$\begin{aligned} \vartheta(z + m) &= \vartheta(z) \\ \vartheta(z + \Omega m) &= \exp(-\pi i \langle \Omega m, m \rangle - 2\pi i \langle z, m \rangle) \vartheta(z) \end{aligned}$$

for all $m \in \mathbb{Z}^n$, as shown in [Mum83]. Given $e \in \mathbb{C}^n$, we rewrite this as $e = u + \Omega v$ for some $u, v \in \mathbb{R}^n$, and we define the *theta function with characteristic e*, $\vartheta[e] : \mathbb{C}^n \rightarrow \mathbb{C}$ by

$$\vartheta[e](z) = \vartheta \left[\begin{array}{c} u \\ v \end{array} \right] (z) = \exp(\pi i \langle \Omega v, v \rangle + 2\pi i \langle z + u, v \rangle) \vartheta(z + e).$$

Note that this follows [Mum83]. Subtly different definitions are used in [Fay73], [DM05] and [FK92], although these differences are not particularly important.

Theorem 4.3. *There exists a constant vector Δ , depending on the choice of base-point, such that for each $e \in \mathbb{C}^n$, either $\vartheta[e] \circ \chi$ is identically zero, or $\vartheta[e] \circ \chi$ has exactly n zeroes, ζ_1, \dots, ζ_n and*

$$\sum_{i=1}^n \chi(\zeta_i) = \Delta - e.$$

Proof. See [Mum83, Ch. 2, Cor. 3.6] or [FK92, VI.2.4]. □

For the following, it will be convenient to define

$$\mathcal{E}_e(x, y) = \vartheta(\chi(y) - \chi(x) + e).$$

Theorem 4.4. *If $e \in \mathbb{C}^n$, $\vartheta(e) = 0$ and \mathcal{E}_e is not identically zero, then there exist $\zeta_1, \dots, \zeta_{n-1}$ such that for each $x \in Y$, $x \neq \zeta_i$, the zeroes of $\vartheta[e - \chi(x)] \circ \chi$, which coincide with the zeroes of $\mathcal{E}_e(x, \cdot)$, are precisely $x, \zeta_1, \dots, \zeta_{n-1}$.*

Proof. See [Mum83, Ch. 2, Lemma 3.4]. □

Theorem 4.5. *There exists an $e_* = u_* + \Omega v_* \in \mathbb{C}^n$ such that $2e_* = 0 \pmod L$, $\langle u_*, v_* \rangle$ is an odd integer, and $\mathcal{E}_{e_*} \neq 0$.*

For the proof see [Mum84, Ch. IIIb, Sec. 1, Lemma 1], although the remarks at the end of [FK92, VI.1.5] provide some relevant discussion. An e_* of this type is called a *non-singular odd half-period*, and we see that $\vartheta[e_*]$ is an odd function, so $\vartheta(e_*) = 0$.

Let $\vartheta_* := \vartheta[e_*]$, so

$$\vartheta_*(t) = \exp(\pi i \langle \Omega v_*, v_* \rangle + 2\pi i \langle z + u_*, v_* \rangle) \vartheta(z + e_*).$$

Clearly, we can apply Theorems 4.4 and 4.5, and get that the roots of

$$\vartheta_*(\chi(\cdot) - \chi(z))$$

are $\{z, \zeta_1, \dots, \zeta_{n-1}\}$ for some $\zeta_1, \dots, \zeta_{n-1}$. If neither of $z, w \in Y$ coincide with any of these ζ_i s, then

$$(4.1) \quad \frac{\vartheta_*(\chi(\cdot) - \chi(z))}{\vartheta_*(\chi(\cdot) - \chi(w))} = e^{2\pi i \langle w - z, v_* \rangle} \frac{\vartheta(\chi(\cdot) - \chi(z) + e_*)}{\vartheta(\chi(\cdot) - \chi(w) + e_*)}$$

is a multiple valued function with exactly one zero and one pole, at z and w respectively.

4.3. The Fay Kernel. A tool that will prove invaluable in later sections is the Fay kernel K^a , which is a reproducing kernel on $\mathbb{H}^2(R, \omega_a)$, the Hardy space of analytic functions on R with boundary values in $L^2(\omega_a)$. For a more comprehensive discussion of the ideas in this section, see [Fay73].

Lemma 4.6. *The critical points of the Green's function $g(\cdot, b)$ are on \mathbb{X} , one in each $\mathbb{X}_i, i = 1, \dots, n$.*

Proof. We write $g(z) = g(z, b)$. We know that g has n critical points, by [Neh52, p. 133-135]. On \mathbb{X} , define $\partial/\partial x$ as the derivative tangent to \mathbb{X} , and $\partial/\partial y$ as the derivative normal to \mathbb{X} . We know that $g \circ \omega = g$, and

$$\frac{\partial g}{\partial y} = \frac{\partial g \circ \omega}{\partial y} = \frac{\partial g}{\partial y} \cdot \frac{\partial \omega_y}{\partial y} + \frac{\partial g}{\partial x} \cdot \frac{\partial \omega_x}{\partial y}.$$

However, $\partial \omega_y / \partial y < 0$ on \mathbb{X} , so the two sides of this equation have different signs, and so $\partial g / \partial y = 0$ on \mathbb{X} . Also, $g = 0$ on B , so g must be zero at p_i^- and p_{i+1}^+ – the start and end points of \mathbb{X}_i . Since $\partial g / \partial x$ is continuous on \mathbb{X}_i (provided $i \neq 0$), $\partial g / \partial x$ must be zero somewhere on \mathbb{X}_i , by Rolle's theorem. Since this gives us n distinct zeroes, this must be all of them. \square

We have just proved that $g(\cdot, b)$ has n distinct zeroes. If these zeroes are $z_1 \in \mathbb{X}_1, \dots, z_n \in \mathbb{X}_n$ we define $P_i = Jz_i$.

Theorem 4.7. *There is a reproducing kernel K^a for the Hardy space $\mathbb{H}^2(R, \omega_a)$; that is, if $f \in \mathbb{H}^2(R, \omega_a)$, then*

$$f(y) = \langle f(\cdot), K^a(\cdot, y) \rangle = \int_{\partial R} f(x) \overline{K^a(x, y)} d\omega_a(x).$$

If $z = a$, then $K^a(\cdot, z) \equiv 1$. If not, $K^a(\cdot, z)$ has precisely the poles

$$P_1(a), \dots, P_n(a), Jz$$

(where $JP_1(a), \dots, JP_n(a)$ are the critical points of $g(\cdot, a)$), and $n + 1$ zeroes in Y , one of which is Ja .

Sketch Proof. By [Fay73, Prop. 6.15], there is an $e \in \mathcal{J}(Y)$ such that

$$(4.2) \quad K^a(x, y) = \frac{\vartheta(\chi(x) + \chi(y)^* + e) \vartheta(\chi(a) + \chi(a)^* + e) \vartheta_*(\chi(a) + \chi(y)^*) \vartheta_*(\chi(x) + \chi(a)^*)}{\vartheta(\chi(a) + \chi(y)^* + e) \vartheta(\chi(x) + \chi(a)^* + e) \vartheta_*(\chi(x) + \chi(y)^*) \vartheta_*(\chi(a) + \chi(a)^*)}$$

is the reproducing kernel² for $\mathbb{H}^2(R, \omega_a)$.

It is clear that $K^a(x, a) = 1$, so we fix a and y and look at the zero/pole structure of $K^a(\cdot, y)$. We can see that for fixed y , the zeroes and poles of (4.2)

²Fay gives (4.2) in a slightly different form, although we can use [Fay73, Prop. 6.1] and some basic results on theta functions to show that the two forms are equivalent. Also note that the notation Fay uses differs significantly from the notation used here.

are precisely the zeroes and poles of

$$\frac{\vartheta(\chi(x) + \chi(y)^* + e) \vartheta_*(\chi(x) + \chi(a)^*)}{\vartheta(\chi(x) + \chi(a)^* + e) \vartheta_*(\chi(x) + \chi(y)^*)}$$

by removing terms with no dependence on x . By (4.1), the ϑ_* factors bring in a zero at Ja and a pole at Jy . The remaining theta functions have n zeroes each, so K^a gets n new poles, $P_1(a), \dots, P_n(a)$, and n new zeroes, $Z_1(y), \dots, Z_n(y)$ from the top and bottom terms respectively. The $P_i(a)$ s must all be in $J(R) \cup B$, as we know that $K^a(\cdot, y)$ is analytic on R .

Suppose, towards a contradiction, that some of these poles and zeroes were to cancel, then $K^a(\cdot, y)$ would have n or fewer poles. If it had no zeroes, it would be constant, but we know that the set $\{K^a(\cdot, y) : y \in R\}$ is linearly independent, and $K^a(\cdot, a)$ is constant, so $K^a(\cdot, y)$ cannot be a multiple of it. If it had one or more poles, then it would be a meromorphic function on Y with between 1 and n poles, all in $J(R) \cup B$. Moreover, Jy cannot cancel with Ja because $a \neq y$, and it cannot cancel with any of the $Z_i(y)$ s since that would mean

$$\begin{aligned} 0 &= \vartheta(\chi(Jy) + \chi(y)^* + e) \\ &= \vartheta(\cancel{-\chi(y)^*} + \chi(y)^* + e) = \vartheta(e), \end{aligned}$$

which Fay shows is not the case, so Jy cannot cancel. We know $Jy \notin B$, so by Proposition 1.4 on page 3, this also leads to a contradiction, and so none of the zeroes and poles cancel. Thus, $K^a(\cdot, y)$ has $n + 1$ zeroes and poles.

We give a sketch proof that the poles are as stated. We use the alternate characterisation of $K^a(x, y)$ given in [Fay73, Prop. 6.15], that is,

$$K^a(x, y) = \left(\frac{\Lambda_a(y, Jx)}{\Omega_{Ja-a}(y)} \right).$$

Note that the notation here is partly that used in Fay, and partly that used in this paper. In particular, Λ and Ω are as defined in Propositions 2.9 and 6.15 of Fay respectively (the definitions are too complicated to replicate here). Clearly, $\Omega_{Ja-a}(y)$ has no dependence on x , so has no direct bearing on the poles in x of K^a . However, we note that the divisor \mathcal{A} used in the construction of Λ is the zero divisor of Ω_{Ja-a} , which is precisely the critical divisor of $g(\cdot, a)$. We then use the description of $\text{div} \Lambda_a$ from [Fay73, Prop. 2.9] to see that for fixed y , the poles of $\Lambda_a(y, J(\cdot))$ are precisely

$$\{Jx\} \cup J(\mathcal{A}) = \{Jx, P_1(a), \dots, P_n(a)\},$$

where the $P_i(a)$ s are as required. □

We will write $P_i(b) = P_i$, for brevity.

Theorem 4.8. *Let a_1^0, \dots, a_{2n}^0 be points in R such that*

$$P_1, \dots, P_n, Jb, Ja_1^0, \dots, Ja_{2n}^0$$

are all distinct. Let $\{e_1, e_2\}$ denote the standard basis for \mathbb{C}^2 and let

$$\gamma_1^0 = \cdots = \gamma_n^0 = e_1, \quad \gamma_{n+1}^0 = \cdots = \gamma_{2n}^0 = e_2.$$

There exists an $\epsilon > 0$ so that if $|a_j^0 - a_j|, \|\gamma_j^0 - \gamma_j\| < \epsilon$, and

$$(4.3) \quad h(z) = \sum_{j=1}^{2n} c_j K^b(z, a_j) \gamma_j + v$$

is a \mathbb{C}^2 -valued meromorphic function which does not have poles at P_1, \dots, P_n , then h is constant; that is, each $c_j = 0$.

Further, if $h \neq 0$ has a representation as in (4.3), and there exists $z' \in R \setminus \{b\}$ such that

$$h(z) K^b(z, z') = \sum c'_j K^b(z, a_j) \gamma_j + v'$$

then h is constant, $z' = a_j$ for some j , $c'_j \gamma_j = h$, and all other terms are zero.

This theorem can be seen as a result about meromorphic functions on Y , so we view z as a local co-ordinate on Y . If we're only interested in values of z near one of P_1, \dots, P_n , we can assume $z, P_1, \dots, P_n, Ja_1, \dots, Ja_{2n}$ are in a single chart $U \subseteq J(R)$ (U is open and simply connected)

A useful tool in the proof of this theorem is the residue of K^b . We know that so long as $a \notin \{b, P_1, \dots, P_n\}$, $K^b(\cdot, a)$ has only simple poles, so we know that in a small enough neighbourhood of P_j ,

$$(z - P_j) K^b(z, a)$$

is a holomorphic function in z . Let $R_j(a)$ denote the value of this function at P_j .

We will need the following lemma.

Lemma 4.9. *The residue $R_j(a)$ varies continuously with a .*

Proof. Consider the theta function representation of $K^b(z, a)$. The function

$$f(z) = \vartheta(\chi(z) + \chi(b)^* + e)$$

is analytic and single valued on U , and vanishes with order one at P_j , so can be written as

$$f(z) = (z - P_j) f_j(z)$$

for some f_j analytic on U , and non-vanishing at P_j . Given a set $W \subseteq U$, let $W^* = \{\bar{z} : z \in W\}$. Choose neighbourhoods V_j, W of U so that $F : V_j \times W^* \rightarrow \mathbb{C}$ given by

$$\begin{aligned} F(z, a) &= f(z) K^b(z, a) \\ &= \frac{\vartheta(\chi(z) + \chi(a)^* + e) \vartheta(\chi(b) + \chi(b)^* + e) \vartheta_*(\chi(b) + \chi(a)^*) \vartheta_*(\chi(z) + \chi(b)^*)}{\vartheta(\chi(b) + \chi(a)^* + e) \vartheta_*(\chi(z) + \chi(a)^*) \vartheta_*(\chi(b) + \chi(b)^*)} \end{aligned}$$

is analytic in (z, \bar{a}) . Rewriting gives

$$(z - P_j)K^b(z, a) = \frac{F(z, a)}{f_j(z)}$$

The lemma follows from the fact that the right hand side is analytic in (z, \bar{a}) . \square

We can now prove Theorem 4.8.

Proof of Theorem 4.8. We can assume ϵ is small enough that

$$P_1, \dots, P_n, Ja_1, \dots, Ja_{2n}$$

are distinct. We define

$$\mathfrak{R}_1 = \begin{pmatrix} R_1(a_1) & \cdots & R_1(a_n) \\ \vdots & \ddots & \vdots \\ R_n(a_1) & \cdots & R_n(a_n) \end{pmatrix}$$

and

$$\mathfrak{R}_2 = \begin{pmatrix} R_1(a_{n+1}) & \cdots & R_1(a_{2n}) \\ \vdots & \ddots & \vdots \\ R_n(a_{n+1}) & \cdots & R_n(a_{2n}) \end{pmatrix},$$

where $R_j(a)$ is the residue of $K^b(\cdot, a)$ at P_j , as before.

To see that \mathfrak{R}_1 is invertible, let

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

and

$$f_c = \sum_{j=1}^n c_j K^b(\cdot, a_j).$$

Note that $\mathfrak{R}_1 c = 0$ if and only if f_c does not have poles at any P_j . Now, if this is the case, then f_c can only have poles at Ja_1, \dots, Ja_n , and simple poles at that, but this is only n points, so by Proposition 1.4, f_c must be constant. We know that $K^b(\cdot, b) = 1$, so we can say that

$$0 = c_0 K^b(\cdot, b) + c_1 K^b(\cdot, a_1) + \cdots + c_n K^b(\cdot, a_n).$$

However, we know that $K^b(\cdot, b), K^b(\cdot, a_1), \dots, K^b(\cdot, a_n)$ are linearly independent, so $c = 0$. Therefore \mathfrak{R}_1 is invertible, and by a similar argument \mathfrak{R}_2 is invertible.

Now, consider the function F defined for γ_j near γ_j^0 by

$$F = \begin{pmatrix} R_1(a_1)\gamma_1 & \cdots & \cdots & R_1(a_{2n})\gamma_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ R_2(a_1)\gamma_1 & \cdots & \cdots & R_2(a_{2n})\gamma_{2n} \end{pmatrix}.$$

We define F_0 similarly, using a_j^0 and γ_j^0 . We can see that F is an $n \times 2n$ matrix with entries from \mathbb{C}^2 , so can be regarded as a $2n \times 2n$ matrix. We know that F varies continuously with each γ_j , and by Lemma 4.9, varies continuously with each a_j . Also, we see that, by regarding F_0 as a $2n \times 2n$ matrix, the rows of F_0 can be shuffled to give

$$\begin{pmatrix} \mathfrak{R}_1 & 0 \\ 0 & \mathfrak{R}_2 \end{pmatrix}$$

which is invertible, so F_0 is invertible. We can therefore choose $\epsilon > 0$ small enough that if $|a_j - a_j^0|, \|\gamma_j - \gamma_j^0\| < \epsilon$ for all j , then F is invertible.

If the a_j and γ_j are chosen such that F is invertible and

$$h(z) = \sum_{j=1}^n c_j K^b(z, a_j) \gamma_j + v$$

does not have poles at P_j , then

$$0 = \begin{pmatrix} \sum_{j=1}^n c_j R_1(a_j) \gamma_j \\ \vdots \\ \sum_{j=1}^n c_j R_n(a_j) \gamma_j \end{pmatrix} = F \begin{pmatrix} c_1 \\ \vdots \\ c_{2n} \end{pmatrix} = Fc,$$

so $c = 0$, and h is constant.

Now we prove the second part of the theorem. Note that the proof of this part only assumes that the result of the first part holds, not the assumptions on a_j and γ_j used to prove it. Suppose $h \neq 0$ and there exists $z' \in R \setminus \{b\}$ such that

$$h(z) K^b(z, z') = \sum c'_j K^b(z, a_j) \gamma_j + v'.$$

We can see that P_1, \dots, P_n are not poles of h , since by the assumptions on the distinctness of the P_k s and a_j s, the right hand side has a pole of order at most one at each P_k , whilst the left hand side has poles of order at least one at each of these points. Therefore, since h has a representation as in the first part of the theorem, h is constant. \square

5. REPRESENTATIONS

This paper inherits much of its structure from [DM05], and in particular, the results in this section are analogues of results from that paper. In fact, in some cases, the proofs in [DM05] do not use the connectivity of X , so can be used to prove their analogues here simply by noting this fact. In these cases, the proofs are omitted.

5.1. Kernels, Realisations and Interpolation. We note, for those who are interested, that many of these results have a similar flavour to some of the Schur-Agler class results from [DM07], although we shall not use any of these results directly.

Lemma 5.1. *If $F \in M_2(\mathbb{H}(X))$, then there exists a $\rho > 0$ such that*

$$I - \rho^2 F(z)F(w)^* \in \mathcal{C}.$$

Theorem 5.2. *If there is a function $F : R \rightarrow M_2(\mathbb{C})$ which is analytic in a neighbourhood of X and unitary valued on B , such that $\rho_F < 1$, then there exists an operator $T \in \mathcal{B}(H)$ for some Hilbert space H , such that the homomorphism $\pi : \mathcal{R}(X) \rightarrow \mathcal{B}(H)$ given by $\pi(p/q) = p(T) \cdot q(T)^{-1}$ is contractive, but not completely contractive.*

Later on in this section, we will need to work with matrix valued Herglotz representations, so we will need some results about matrix-valued measures. Given a compact Hausdorff space X , an $m \times m$ matrix-valued measure

$$\mu = \left(\mu_{jl} \right)_{j,l=1}^m$$

is an $m \times m$ matrix whose entries μ_{jl} are complex-valued Borel measures on X . The measure μ is positive (we write $\mu \geq 0$) if for each function $f : X \rightarrow \mathbb{C}^m$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix},$$

we have

$$0 \leq \sum_{j,l} \int_X \bar{f}_j f_l d\mu_{jl}.$$

The positive measure μ is bounded by $M > 0$ if

$$MI_m - \left(\mu_{jl}(X) \right) \geq 0$$

is positive semi-definite, where I_m is the $m \times m$ identity matrix.

Lemma 5.3. *The $m \times m$, matrix-valued measure μ is positive if and only if for each Borel set ω the $m \times m$ matrix*

$$\left(\mu_{jl}(\omega) \right)$$

is positive semi-definite.

Further, if there is a κ so that each diagonal entry $\mu_{jj}(X) \leq \kappa$, then each entry μ_{jl} of μ has total variation at most κ . Particularly, if μ is bounded by M , then each entry has variation at most M .

Lemma 5.4. *If μ^n is a sequence of positive $m \times m$ matrix-valued measures on X which are all bounded above by M , then μ^n has a weak-* convergent sub-sequence, that is, there exists a positive $m \times m$ matrix-valued measure μ , such that for each pair of continuous functions $f, g : X \rightarrow \mathbb{C}^m$,*

$$\sum_{j,l} \int_X f_l \bar{g}_j d\mu_{jl}^{n_k} \rightarrow \sum_{j,l} \int_X f_l \bar{g}_j d\mu_{jl}.$$

Lemma 5.5. *If μ is a positive $m \times m$ matrix-valued measure on X , then the diagonal entries, μ_{jj} are positive measures. Further, with $\nu = \sum_j \mu_{jj}$, there exists an $m \times m$ matrix-valued function $\Delta : X \rightarrow M_m(\mathbb{C})$ so that $\Delta(x)$ is positive semi-definite for each $x \in X$ and $d\mu = \Delta d\nu$ – that is, for each pair of continuous functions, $f, g : X \rightarrow \mathbb{C}^m$,*

$$\sum_{j,l} \int_X \bar{g}_j f_l d\mu_{jl} = \sum_{j,l} \int_X \bar{g}_j \Delta_{jl} f_l d\nu.$$

A key result of this section is the existence of a Herglotz representation for well behaved inner functions, as follows.

Proposition 5.6. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R , F is unitary valued on B , and $F(b) = 0$. If $\rho_F = 1$ and if $S \subseteq R$ is a finite set, then there exists a probability measure μ on Π and a positive kernel $\Gamma : S \times S \times \Pi \rightarrow \mathbb{C}$ so that*

$$1 - F(z)F(w)^* = \int_{\Pi} \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) \Gamma(z, w; p) d\mu(p).$$

Proof. The proof of this result is almost identical to that of [DM05, Prop. 5.6], except that functions required to vanish at zero, are now required to vanish at b instead. \square

Another tool that will prove useful is transfer function representations. For our purposes it will suffice to work with relatively simple colligations. We will define a unitary colligation Σ by $\Sigma = (U, K, \mu)$, where μ is a probability measure on Π , K is a Hilbert space, and U is a linear operator, defined by

$$U = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \mathcal{B} \left(\begin{array}{c} L^2(\mu) \otimes K \\ \oplus \\ \mathbb{C}^2 \end{array} \right),$$

where $L^2 \otimes K$ can be regarded as K valued L^2 .

We define $\Phi : R \rightarrow \mathcal{B}(L^2(\mu) \otimes K)$ by

$$(\Phi(z) f)(p) = \psi_p(z) f(p).$$

From here, we define the transfer function associated to Σ by

$$W_{\Sigma}(z) = \mathbf{D} + \mathbf{C}\Phi(z)(I - \Phi(z)\mathbf{A})^{-1}\Phi(z)\mathbf{B}.$$

We can see that as \mathbf{A} is a contraction and $\Phi(z)$ is a strict contraction, the inverse in W_{Σ} exists for any $z \in R$.

Proposition 5.7. *The transfer function is contraction valued, that is, $\|W_{\Sigma}(z)\| \leq 1$ for all $z \in R$. In fact for all $z, w \in R$*

$$I - W_{\Sigma}(z)W_{\Sigma}(w)^* = \mathbf{C}(I - \Phi(z)\mathbf{A})^{-1} [I - \Phi(z)\Phi(w)^*] (I - \Phi(w)\mathbf{A})^{*-1} \mathbf{C}^*.$$

Note that if we define $H(w) = (I - \mathbf{A}^*\Phi(w)^*)^{-1} \mathbf{C}^*$, for w fixed, $H(w)^*$ is a function on Π , so we write $H_p(w)^*$. We can see that by considering $L^2(\mu) \otimes K$

as a measure space, Proposition 5.7 on the previous page gives

$$I - W(z)W(w)^* = \int_{\Pi} \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) H_p(z)H_p(w)^* d\mu(p).$$

Proposition 5.8. *If $S \subseteq R$ is a finite set, $W : S \rightarrow M_2(\mathbb{C})$ and there is a positive kernel $\Gamma : S \times S \times \Pi \rightarrow M_2(\mathbb{C})$ such that*

$$I - W(z)W(w)^* = \int_{\Pi} \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) \Gamma(z, w; p) d\mu(p)$$

for all $z, w \in S$, then there exists $G : R \rightarrow M_2(\mathbb{C})$ such that G is analytic, $\|G(z)\| \leq 1$ and $G(z) = W(z)$ for $z \in S$. Indeed, there exists a finite-dimensional Hilbert space K (dimension at most $2|S|$) and a unitary colligation $\Sigma = (U, K, \mu)$ so that

$$G = W_{\Sigma},$$

and hence there exists $\Delta : R \times R \times \Pi \rightarrow M_2(\mathbb{C})$ a positive analytic kernel such that

$$I - G(z)G(w)^* = \int_{\Pi} \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) \Delta(z, w; p) d\mu(p)$$

for all $z, w \in R$.

The proof is as in [DM05], although for our purposes it makes sense to use the version of Kolmogorov's theorem in [AM02, Thm. 2.62].

5.2. Uniqueness.

Proposition 5.9. *Suppose $F : R \rightarrow M_2(\mathbb{C})$ is analytic in a neighbourhood of X , unitary on B , and with a standard zero set. Then there exists a set $S \subseteq R$ with $2n + 3$ elements such that, if $Z : R \rightarrow M_2(\mathbb{C})$ is contraction-valued, analytic, and $Z(z) = F(z)$ for $z \in S$, then $Z = F$.*

Proof. Let K^b denote the Fay kernel for R defined in Theorem 4.7 on page 23. That is, K^b is the reproducing kernel for the Hilbert space

$$\mathbb{H}^2 := \mathbb{H}^2(R, \omega_b)$$

of functions analytic in R with $L^2(\omega_b)$ boundary values. Let \mathbb{H}_2^2 denote \mathbb{C}^2 -valued \mathbb{H}^2 . Since F is unitary valued on B , the mapping V on \mathbb{H}_2^2 given by $VG(z) = F(z)G(z)$ is an isometry. Also, as we will show, the kernel of V^* is the span of

$$\mathfrak{B} := \left\{ K^b(\cdot, a_j)\gamma_j : j = 1, \dots, 2n + 2 \right\},$$

where $F(a_j)^*\gamma_j = 0$ and $\gamma_j \neq 0$; that is, (a_j, γ_j) is a zero of F^* .

We note, for future use, that if φ is a scalar-valued analytic function on a neighbourhood of R , with no zeroes on B , and zeroes $w_1, \dots, w_n \in R$, all of multiplicity one, and $f \in \mathbb{H}^2$ has roots at all these w_i s, then $f = \varphi g$ for some $g \in \mathbb{H}^2$.

Now, suppose $\psi \in \mathbb{H}^2$ and for all $h \in \mathbb{H}^2$ we have $\langle \psi, \varphi h \rangle = 0$. Since the set

$$\mathfrak{R} := \{K^b(\cdot, w_j) : 1 \leq j \leq n\}$$

is linearly independent, we know there is some linear combination

$$f = \psi - \sum_{j=1}^n c_j K^b(\cdot, w_j),$$

so that $f(w_j) = 0$ for all j , and so $f = \varphi g$ for some g . Since

$$\langle K^b(\cdot, w_j), \varphi h \rangle = \overline{\varphi(w_j)} h(w_j) = 0$$

for each j and h , it follows that $\langle f, \varphi h \rangle = 0$ for all h . In particular, if $h = g$ (the g we found earlier), then

$$\langle \varphi g, \varphi g \rangle = \langle f, \varphi g \rangle = 0,$$

so $g \equiv 0$, and so

$$(5.1) \quad 0 = f = \psi - \sum_{j=1}^n c_j K^b(\cdot, w_j).$$

This tells us that ψ is in the span of \mathfrak{R} , so \mathfrak{R} is a basis for the orthogonal complement of $\{\varphi h : h \in \mathbb{H}^2\}$.

We now find the kernel of V^* . Write $a_{2n+1} = a_{2n+2} = b$. Since $F(b) = 0$, there is a function H analytic in a neighbourhood of X so that $F(z) = (z - b)H(z)$. The function $\varphi(z) = (z - b) \det(H(z))$ satisfies the hypothesis of the preceding paragraph.

Let

$$G := \begin{pmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{pmatrix},$$

where $H = (h_{jl})$. Then

$$FG = (z - b)HG = (z - b) \det(H)I,$$

where I is the 2×2 identity matrix.

Now, suppose $x \in \mathbb{H}_2^2$ and $V^*x = 0$. Let x_1, x_2 be the co-ordinates of x . For each $g \in \mathbb{H}_2^2$,

$$\begin{aligned} 0 &= \langle Gg, V^*x \rangle \\ &= \langle VGg, x \rangle \\ &= \langle (z - b) \det(H)g, x \rangle \\ &= \langle (z - b) \det(H)g_1, x_1 \rangle + \langle (z - b) \det(H)g_2, x_2 \rangle. \end{aligned}$$

It therefore follows from the discussion leading up to (5.1) that both x_1 and x_2 are in the span of

$$\{K^b(\cdot, a_j) : 1 \leq j \leq 2n + 2\},$$

so

$$x \in \text{Span} \left\{ K^b(\cdot, a_j)v : 1 \leq j \leq 2n+2, v \in \mathbb{C}^2 \right\}.$$

In particular, there exist vectors $v_j \in \mathbb{C}^2$ such that

$$x = \sum_{j=1}^{2n+2} K^b(\cdot, a_j)v_j.$$

We can check that $V^*vK^b(\cdot, a) = F(a)^*vK^b(\cdot, a)$, and $F(b)^* = 0$, so

$$0 = V^*x = \sum_{j=1}^{2n+2} F(a_j)^*v_jK^b(\cdot, a_j),$$

but the $K^b(\cdot, a_j)$ s are linearly independent, so $F(a_j)^*v_j = 0$ for all j . Conversely, if $F(a_j)^*v_j = 0$ then $V^*v_jK^b(\cdot, a_j) = 0$, so the kernel of V^* is spanned by \mathfrak{B} .

Now, since V is an isometry, $I - VV^*$ is the projection onto the kernel of V^* , which by the above argument has dimension $2n+2$, so $I - VV^*$ has rank $2n+2$. So, for any finite set $A \subseteq R$, the block matrix with 2×2 entries

$$\begin{aligned} M_A &= \left(\left[\left\langle (I - VV^*)K^b(\cdot, w)e_j, K^b(\cdot, z)e_l \right\rangle \right]_{j,l=1,2} \right)_{z,w \in A} \\ &= \left((I - F(z)F(w)^*)K^b(z, w) \right)_{z,w \in A} \end{aligned}$$

has rank at most $2n+2$. In particular, if $A = \{a_1, \dots, a_{2n+2}\}$, then M_A has rank exactly $2n+2$. Choose a_{2n+3}, a_{2n+4} distinct from a_1, \dots, a_{2n+2} so that

$$S = \{a_1, \dots, a_{2n+2}, a_{2n+3}, a_{2n+4}\}$$

has $2n+3$ distinct points. Since $A \subseteq S$, M_S has rank at least $2n+2$. However, by the above discussion, its rank cannot exceed $2n+2$, so its rank must be exactly $2n+2$.

The matrix M_S is $(4n+6) \times (4n+6)$, (a $(2n+3) \times (2n+3)$ matrix with 2×2 matrices as its entries), and M_S has rank $2n+2$, so must have nullity (that is, kernel dimension) $2n+4$. Further, the subspace

$$\mathcal{L}_1 := \left\{ \underbrace{\begin{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \\ \vdots \\ \alpha_{2n+3} \\ 0 \end{pmatrix} \\ = \alpha \otimes e_1 \end{pmatrix}} : \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n+3} \end{pmatrix} \in \mathbb{C}^{2n+3} \right\}$$

is $2n+3$ dimensional, so there exists a non-zero $x_1 = y_1 \otimes e_1$ in \mathcal{L}_1 which is in the kernel of M_S . Similarly, $\mathcal{L}_2 := \{\alpha \otimes e_2 : \alpha \in \mathbb{C}^{2n+3}\}$ contains some x_2 in the kernel of M_S .

Let $x = (x_1 \ x_2)$, so x is the $(4n + 6) \times 2$ matrix

$$x = \begin{pmatrix} \begin{pmatrix} (y_1)_1 & 0 \\ 0 & (y_2)_1 \end{pmatrix} \\ \begin{pmatrix} (y_1)_2 & 0 \\ 0 & (y_2)_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} (y_1)_{2n+3} & 0 \\ 0 & (y_2)_{2n+3} \end{pmatrix} \end{pmatrix}.$$

It will be more convenient to refer to 2×2 blocks in x by their corresponding point in S , rather than their number, so we say

$$x(w) = (x_1(w) \ x_2(w)) = \begin{pmatrix} y_1(w) & 0 \\ 0 & y_2(w) \end{pmatrix}.$$

In this notation, the identity $M_S x = 0$ becomes

$$\sum_{w \in S} K^b(z, w) x(w) = F(z) \sum_{w \in S} K^b(z, w) F(w)^* x(w)$$

for each z .

Now, suppose $Z : R \rightarrow M_2(\mathbb{C})$ is analytic, contraction valued, and $Z(z) = F(z)$ for $z \in S$. The operator W of multiplication by Z on \mathbb{H}_2^2 is a contraction and

$$W^* K^b(\cdot, w) v = Z(w)^* v K^b(\cdot, w).$$

Given $\zeta \in R$, $\zeta \notin S$, let $S' = S \cup \{\zeta\}$ and consider the decomposition of

$$N_\zeta = \left((I - Z(z)Z(w)^*) K^b(z, w) \right)_{z, w \in S'}$$

into blocks labelled by S and $\{\zeta\}$. Thus N_ζ is a $(2n + 4) \times (2n + 4)$ matrix with 2×2 block entries. The upper left $(2n + 3) \times (2n + 3)$ block is simply M_S , as $Z(z) = F(z)$ for $z \in S$.

Let

$$x' = \begin{pmatrix} x \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Since N_ζ is positive semi-definite and $M_S x = 0$, it can be shown that $N_\zeta x' = 0$.

An examination of the last two entries of the equation $N_\zeta x' = 0$ gives

$$(5.2) \quad \sum_{w \in S} K^b(\zeta, w) x(w) = Z(\zeta) \sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w).$$

The left hand side of (5.2) is a rank 2, 2×2 matrix at all but countably many ζ , as it is a diagonal matrix whose diagonal elements are of the form

$$\sum_{w \in S} K^b(\zeta, w) y_i(w);$$

that is, linear combinations of $K^b(\zeta, w)$ s. If such a function is zero at an uncountable number of ζ s, it is identically zero, which is impossible, as the $K^b(\cdot, w)$ s are linearly independent and the $y_i(w)$ s are not all zero. We can now see that

$$\sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w)$$

is invertible at all but countably many ζ , so

$$\begin{aligned} Z(\zeta) &= \sum_{w \in S} K^b(\zeta, w) x(w) \left(\sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w) \right)^{-1} \\ &= \sum_{w \in S} K^b(\zeta, w) x(w) \left(\sum_{w \in S} F(w)^* K^b(\zeta, w) x(w) \right)^{-1} \\ &= F(\zeta) \end{aligned}$$

at all but finitely many ζ , so $Z = F$. \square

We combine some of the preceding results to get the following.

Theorem 5.10. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R , which is unitary-valued on B , and with a standard zero set. If $\rho_F = 1$, then there exists a unitary colligation $\Sigma = (U, K, \mu)$ such that $F = W_\Sigma$, and so that the dimension of K is at most $4n + 6$. In particular, μ is a probability measure on Π and there is an analytic function $H : R \rightarrow L^2(\mu) \otimes M_{4n+6, 2}(\mathbb{C})$, denoted by $H_p(z)$, so that*

$$I - F(z)F(w)^* = \int_{\Pi} (1 - \psi_p(z)\overline{\psi_p(w)}) H_p(z)H_p(w)^* d\mu(p)$$

for all $z, w \in R$.

Proof. Using Proposition 5.9 on page 30, choose a finite set $S \subseteq R$ such that if $G : R \rightarrow M_2(\mathbb{C})$ is analytic and contraction valued, and $G(z) = F(z)$ for $z \in S$, then $G = F$. Using Proposition 5.6 on page 29, we have a probability measure μ and a positive kernel $\Gamma : S \times S \times \Pi \rightarrow M_2(\mathbb{C})$ such that

$$I - F(z)F(w)^* = \int_{\Pi} (1 - \psi_p(z)\overline{\psi_p(w)}) \Gamma(z, w; p) d\mu(p)$$

for all $z, w \in S$.

By Proposition 5.8 on page 30, there exists a unitary colligation $\Sigma = (U, K, \mu)$ so that K is at most $4n + 6$ dimensional, and $W_\Sigma(z) = F(z)$ for $z \in S$. However, our choice of S gives $W_\Sigma = F$ everywhere. We know $\Gamma(z, w; p) = H_p(z)H_p(w)^*$ for some H_p by [AM02, Thm. 2.62]. \square

Theorem 5.11. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R , which is unitary valued on B , with a standard zero set, and $\rho_F = 1$, and is represented as in Theorem 5.10. Let $a_{2n+1} = a_{2n+2} = b$, $\gamma_{2n+1} = e_1$, and $\gamma_{2n+2} = e_2$. Then there exists a set E of μ measure zero, such that for $p \notin E$, for*

each $v \in \mathbb{C}^{4n+6}$, and for $l = 0, 1, \dots, n$, the vector function $H_p(\cdot)vK^b(\cdot, z_l)$ is in the span of $\{K^b(\cdot, a_j)\gamma_j\}$, where $z_0(p) (= b)$, $z_1(p), \dots, z_n(p)$ are the zeroes of ψ_p . Consequently, H_p is analytic on R and extends to a meromorphic function on Y .

Proof. We showed in Proposition 5.9 on page 30 that given a finite $Q \subseteq R$,

$$M_Q = \left((I - F(z)F(w)^*) K^b(z, w) \right)_{z, w \in Q}$$

has rank at most $2n + 2$, and that the range of M_Q lies in

$$(5.3) \quad \mathfrak{M} := \text{span} \left\{ \left(K^b(z, a_i)\gamma_i \right)_{z \in Q} : i = 1, \dots, 2n + 2 \right\},$$

thinking of $\left(K^b(z, a_i)\gamma_i \right)_{z \in Q}$ as a column vector indexed by Q .

We then apply Theorem 5.10 on the facing page to give

$$M_Q = \left(\int_{\Pi} H_p(z) \left(1 - \psi_p(z)\overline{\psi_p(w)} \right) K^b(z, w) H_p(w)^* d\mu(p) \right)_{z, w \in Q}.$$

For each p , we define an operator $M_p \in \mathcal{B}(\mathbb{H}^2)$ by

$$(M_p f)(x) = \psi_p(x) f(x).$$

Multiplication by ψ_p is isometric on \mathbb{H}^2 , so $1 - M_p M_p^* \geq 0$, and so $(1 - M_p M_p^*) \otimes E \geq 0$, where E is the $m \times m$ matrix with all entries equal to 1. From the reproducing property of K^b , we see that $M_p^* K^b(\cdot, z) = \overline{\psi_p(z)} K^b(\cdot, z)$. Thus, if Q is a set of m points in R , and c is the vector $\left(K^0(\cdot, w) \right)_{w \in Q}$, then the matrix

$$P_Q(p) = \left\langle \left[(I - M_p M_p^*) \otimes E \right] c, c \right\rangle = \left\langle \left[1 - \psi_p(z)\overline{\psi_p(w)} \right] K^b(z, w) \right\rangle_{z, w \in Q} \geq 0.$$

If we set $\tilde{Q} = Q \cup \{z_j\}$ for any $j = 0, 1, \dots, n$, then $P_{\tilde{Q}}(p) \geq 0$. Further, the upper $m \times m$ block equals $P_Q(p)$ and the right $m \times 1$ column is $\left(K^b(z, z_j(p)) \right)_{z \in Q}$. Hence, as a vector,

$$\left(K^b(z, z_j(p)) \right)_{z \in Q} \in \text{ran} P_Q(p)^{1/2} = \text{ran} P_Q(p),$$

for $j = 0, 1, \dots, n$.

Since $P_Q \geq 0$,

$$N_Q(p) := \left(H_p(z) \left(1 - \psi_p(z)\overline{\psi_p(w)} \right) K^b(z, w) H_p(w)^* \right)_{z, w \in Q}$$

is also positive semi-definite for each p . If $M_Q x = 0$, then

$$0 = \int_{\Pi} \langle N_Q(p)x, x \rangle d\mu(p),$$

so that $\langle N_Q(p)x, x \rangle = 0$ for almost all p . It follows that $N_Q(p)x = 0$ almost everywhere. Choosing a basis for the kernel of M_Q , there is a set E_Q of μ measure zero so that for $p \notin E_Q$, the kernel of M_Q is a subspace of the kernel of $N_Q(p)$. For such p , the range of $N_Q(p)$ is a subspace of the range of M_Q , so the rank of $N_Q(p)$ is at most $2n + 2$.

Further, if we let $D_Q(p)$ denote the diagonal matrix with $(2 \times (4n+6))$ block entries given by

$$D_Q(p)_{z,w} = \begin{cases} H_p(z) & z = w \\ 0 & z \neq w \end{cases}.$$

then $N_Q(p) = D_Q(p)P_Q(p)D_Q(p)^*$. Since $P_Q(p)$ is positive semi-definite, we conclude that the range of $D_Q(p)P_Q(p)$ is in the range of M_Q . Therefore, since $(K^b(z, z_j(p)))_{z \in Q}$ is in the range of $P_Q(p)$, $(H_p(z) \nu K^b(z, z_j(p)))_{z \in Q}$ is in the range of M_Q for every $\nu \in \mathbb{C}^{4n+6}$, and $j = 0, 1, \dots, n$.

Now suppose $Q_m \subseteq R$ is a finite set with

$$Q_m \subseteq Q_{m+1}, \quad Q_0 = \{a_1, \dots, a_{2n}, a_{2n+1}(=b)\},$$

and

$$\mathcal{D} = \bigcup_{m \in \mathbb{N}} Q_m$$

a determining set; that is, an analytic function is uniquely determined by its values on \mathcal{D} . Since

$$(H_p(z) \nu K^b(z, z_j(p)))_{z \in Q_m} \in \text{ran} M_{Q_m} \subseteq \mathfrak{M},$$

we see that there are constants $c_i^m(p)$ such that

$$(5.4) \quad H_p(z) \nu K^b(z, z_j(p)) = \sum_{i=1}^{2n+2} c_i^m(p) K^b(z, a_i) \gamma_i, \quad z \in Q_m.$$

By linear independence of the $K^b(\cdot, a_i)$ s, the $c_i^m(p)$ s are uniquely determined when $n = 0, 1, \dots$ by this formula. Since $Q_{m+1} \supseteq Q_m$, we see that $c_i^{m+1}(p) = c_i^m(p)$ for all m , so there are unique constants $c_i(p)$ such that

$$H_p(z) \nu K^b(z, z_j(p)) = \sum_{i=1}^{2n+2} c_i(p) K^b(z, a_i) \gamma_i, \quad z \in \mathcal{D}.$$

Now, by considering this equation when $j = 0$, and using the fact that $K^b(\cdot, b) \equiv 1$, we see that H_p agrees with an analytic function on a determining set. We can therefore assume that H_p is analytic for each $p \notin E$, and that (5.4) holds throughout R . Also, since the $K^b(\cdot, a_i)$ s extend to meromorphic functions on Y , so must H_p . \square

5.3. Diagonalisation.

Lemma 5.12. *Suppose F is a matrix-valued function on R whose determinant is not identically zero. If there exists a 2×2 unitary matrix U and scalar valued functions $\phi_1, \phi_2 : R \rightarrow \mathbb{C}$ such that $F(z)F(w)^* = UD(z)D(w)^*U^*$, where*

$$D := \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix},$$

then there exists a unitary matrix V such that $F = UDV$.

Proof. The proof is as in [DM05]. We let $V = D(z)^{-1}U^*F(z)$, which turns out to be constant and unitary. \square

Theorem 5.13. *Suppose F is a 2×2 matrix-valued function which is analytic in a neighbourhood of R , unitary valued on B , and has a standard zero set (a_j, γ_j) , $j = 1, \dots, 2n$. Assume further that the (a_j, γ_j) have the property that if h satisfies*

$$h = \sum_{j=1}^{2n} c_j K^b(\cdot, a_j) \gamma_j + v,$$

for some $c_1, \dots, c_{2n} \in \mathbb{C}$ and $v \in \mathbb{C}^2$, and h does not have a pole at P_1, \dots, P_n , then h is constant.

Under these conditions, if $\rho_F = 1$, then F is diagonalisable, that is, there exists unitary 2×2 matrices U , and V and analytic functions $\phi_1, \phi_2 : R \rightarrow \mathbb{C}$ such that

$$F = U \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} V = UDV.$$

Proof. By Theorem 5.11 on page 34, we may assume that except on a set E of measure zero, if h is a column of some H_p , then $h(\cdot)K^b(\cdot, z_l(s)) \in \mathfrak{M}$ for $l = 0, 1, \dots, n$.³

By hypothesis, h (and so H_p) is constant. From Remark 3.2 on page 16, we can assume at least one of the zeroes of ψ_p (say $z_1(p)$) is not b . Thus, using the proof of the second part of Theorem 4.8 on page 24, we can show that if h is not zero, then $z_1(p) = a_{j_1(p)}$ for some $j_1(p)$, and h is a multiple of $\gamma_{j_1(p)}$. Thus, every column of H_p is a multiple of $\gamma_{j_1(p)}$.

Theorem 5.10 on page 34 gives us

$$I - F(z)F(w)^* = \int_{\Pi} (1 - \psi_p(z)\overline{\psi_p(w)}) H_p H_p^* d\mu(p),$$

and substituting $w = b$ gives

$$I = \int_{\Pi} H_p H_p^* d\mu(p)$$

so

$$(5.5) \quad F(z)F(w)^* = \int_{\Pi} \psi_p(z)\overline{\psi_p(w)} H_p H_p^* d\mu(p).$$

Since the columns of H_p are all multiples of $\gamma_{j_1(p)}$, $H_p H_p^*$ is rank one, and so can be written as $G(p)G(p)^*$ for a single vector $G(p) \in \mathbb{C}^2$. Consequently,

$$(5.6) \quad F(z)F(w)^* = \int_{\Pi} \psi_p(z)\overline{\psi_p(w)} G(p)G(p)^* d\mu(p).$$

Since $F(a_j)^* \gamma_j = 0$ for all j , (5.6) gives

$$0 = \gamma_j^* F(a_j) F(a_j)^* \gamma_j = \int_{\Pi} |\psi_p(a_j)|^2 \|G(p)\gamma_j^*\|^2 d\mu(p),$$

³Here $z_0(p) = b, z_1(p), \dots, z_n(p)$ are the zeroes of ψ_p , and \mathfrak{M} is as defined in (5.3) on page 35.

so for each j , $\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$ for almost every p . So, apart from a set $Z_0 \subseteq \Pi$ of measure zero, $\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$ for all p and all j . Thus, by defining $G(p) = 0$ for $p \in Z_0$, we can assume that (5.6) holds and

$$\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$$

for all values of p and j .

Let $\Pi_0 := \{p \in \Pi : G(p) = 0\}$. If $p \notin \Pi_0$, then for each j , either $\psi_p(a_j) = 0$ or $G(p)^*\gamma_j = 0$. Remember that G_p is a multiple of $\gamma_{j_1(p)}$, and no set of $n+1$ of the γ_j all lie on the same line through the origin. It follows that ψ_p has zeroes at b , and n of the a_j s (say $a_{j_1(p)}, \dots, a_{j_n(p)}$) and $G(p)^*\gamma_j = 0$ at n of the γ_j s (say $\gamma_{j_{n+1}(p)}, \dots, \gamma_{j_{2n}(p)}$), so these γ_j s must be orthogonal to $\gamma_{j_1(p)}$, and so all lie on the same line through the origin. This tells us that the zeroes of ψ_p are precisely $b, a_{j_1(p)}, \dots, a_{j_n(p)}$, so $z_i = a_{j_i(p)}$ for all i . We can also see that $\gamma_{j_1(p)}, \dots, \gamma_{j_n(p)}$ all lie on the same line through the origin, and so are orthogonal to $\gamma_{j_{n+1}(p)}, \dots, \gamma_{j_{2n}(p)}$.

Let $\mathfrak{S}_1 = \{a_{j_1(p)}, \dots, a_{j_n(p)}\}$, $\mathfrak{S}_2 = \{a_{j_{n+1}(p)}, \dots, a_{j_{2n}(p)}\}$, let \mathfrak{A}_1 denote the one-dimensional subspace of \mathbb{C}^2 spanned by $\gamma_{j_1(p)}$ and \mathfrak{A}_2 denote the one-dimensional space spanned by $\gamma_{j_{n+1}(p)}$.

If $q \notin \Pi_0$, then by arguing as above, either $G(q) \in \mathfrak{A}_1$ or $G(q) \in \mathfrak{A}_2$, and the zeroes of ψ_q are in \mathfrak{S}_2 or \mathfrak{S}_1 respectively. Hence, for each p , one of the following must hold:

- (0): $G(p) = 0$;
- (1): $G(p) \in \mathfrak{A}_1$ and the zeroes of ψ_q are in $\mathfrak{S}_2 \cup \{b\}$;
- (2): $G(p) \in \mathfrak{A}_2$ and the zeroes of ψ_q are in $\mathfrak{S}_1 \cup \{b\}$.

Define

$$\Pi_0 = \{p \in \Pi : (0) \text{ holds}\},$$

$$\Pi_1 = \{p \in \Pi : (1) \text{ holds}\},$$

$$\Pi_2 = \{p \in \Pi : (2) \text{ holds}\}.$$

If $p, q \in \Pi_1$ then ψ_p and ψ_q are equal, up to multiplication by a unimodular constant, so we choose a $p^1 \in \Pi_1$ and define $\psi_1 = \psi_{p^1}$, so $\overline{\psi_p \psi_p} = \overline{\psi_1 \psi_1}$ for all $p \in \Pi_1$. If Π_2 is non-empty, we do the same, if not we define $\psi_2 \equiv 0$. We substitute this into (5.5) to get

$$F(z)F(w)^* = h_1 \psi_1(z) \overline{\psi_1(w)} h_1^* + h_2 \psi_2(z) \overline{\psi_2(w)} h_2^*,$$

where $h_j \in \mathfrak{A}_j$. Letting $z = w \in B$, we see that h_1, h_2 is an orthonormal basis for \mathbb{C}^2 (and that $\psi_2 \not\equiv 0$), so we can apply Lemma 5.12 on page 36, and the result follows. \square

6. THE COUNTEREXAMPLE

We now have all the tools we need to prove Theorem 0.2, as introduced at the beginning of the paper. First, we constructed $\Psi_{S,\mathbf{p}}$ in Lemma 3.7 on page 18, which is always a 2×2 matrix-valued inner function. We then showed, in Lemma 3.8 on page 19, that there was a sequence $\Psi_{S_m,\mathbf{p}}$, such that each term had a standard zero set, with $S_m \neq S_0$ for all m , and such that both $S_m \rightarrow S_0$ and $\Psi_{S_m,\mathbf{p}} \rightarrow \Psi_{S_0,\mathbf{p}}$ as $m \rightarrow \infty$. We showed in Theorem 4.8 on page 24, that if the zeroes (a_j, γ_j) of $\Psi_{S_m,\mathbf{p}}$ are close enough to the zeroes of $\Psi_{S_0,\mathbf{p}}$ (they would be, for m large enough, say $m = \mathbf{M}$) then any \mathbb{C}^2 -valued meromorphic function of the form

$$h(z) = \sum_{j=1}^{2n} c_j K^b(z, a_j) \gamma_j + v$$

with no poles at P_1, \dots, P_n must be constant. Thus, we take $\Psi = \Psi_{S_{\mathbf{M}},\mathbf{p}}$. Theorem 5.13 on page 37 then tells us that if $\rho_\Psi = 1$, then Ψ is diagonalisable. So if Ψ is not diagonalisable, then $\rho_\Psi < 1$. If $\rho_\Psi < 1$, Theorem 5.2 on page 28 tells us that there is an operator $T \in \mathcal{B}(H)$ for some H , such that the homomorphism $\pi : \mathcal{R}(X) \rightarrow \mathcal{B}(H)$ with $\pi(p/q) = p(T) \cdot q(T)^{-1}$ is contractive but not completely contractive. Therefore, all that remains to be shown is that Ψ is not diagonalisable.

Theorem 6.1. *Ψ is not diagonalisable.*

Proof. Suppose, towards an eventual contradiction, that there is a diagonal function D and fixed unitaries U and V such that $D(z) = U\Psi(z)V^*$. D must be unitary valued on B , so must be unitary valued at p_0^- , so by multiplying on the left by $D(p_0^-)^*$, we may assume that $D(p_0^-) = I$. Since $\Psi(p_0^-) = I$, $U = V$.

Let

$$D = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}.$$

Since D is unitary on B , both ϕ_1 and ϕ_2 are unimodular on B . Further, as $\det \Psi$ has $2n + 2$ zeroes (up to multiplicity), and a non-constant scalar inner function has at least $n + 1$ zeroes, we conclude that either ϕ_1 and ϕ_2 have $n + 1$ zeroes each, and take each value in the unit disc \mathbb{D} at least $n + 1$ times, or one has $2n + 2$ zeroes, and the other is a unimodular constant λ . The latter cannot occur, since

$$0 = \Psi(b) = U^* \begin{pmatrix} \lambda & \cdot \\ \cdot & \cdot \end{pmatrix} U \neq 0,$$

which would be a contradiction.

Now, from Lemma 3.7 on page 18, $\Psi(\mathbf{p}_1)e_1 = e_1$, so Ue_1 is an eigenvector of $D(\mathbf{p}_1)$, corresponding to the eigenvalue 1, so at least one of the $\phi_j(\mathbf{p}_1)$ s is equal to 1. Similarly, Ue_2 is an eigenvector of $D(\omega(\mathbf{p}_1))$, so at least one of the

$\phi_j(\varpi(\mathbf{p}_1))$ s is equal to 1. Now, $D(\mathbf{p}_1)$ cannot be a multiple of the identity, as this would mean that one of the ϕ_j s was equal to 1 at \mathbf{p}_1 and $\varpi(\mathbf{p}_1)$, which is impossible⁴. Therefore, we can assume without loss of generality that

$$D(\mathbf{p}_1) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad D(\varpi(\mathbf{p}_1)) = \begin{pmatrix} \lambda' & 0 \\ 0 & 1 \end{pmatrix},$$

where λ, λ' are unimodular constants. We can see from this that the eigenvectors corresponding to 1 in these matrices are e_1 and e_2 , so $Ue_1 = ue_1$, $Ue_2 = u'e_2$ for unimodular constants u, u' . Since D is diagonal, we can assume that $u = u' = 1$, so $U = I$, and $\Psi = D$.

Now, since $S_M \neq S_0$, there exists some i such that $P^{i+} \neq P^{1+}$, so these two projections must have different ranges. However by Lemma 3.7,

$$\begin{aligned} P^{i+} &= \Psi(\mathbf{p}_i) P^{i+} \\ &= D(\mathbf{p}_i) P^{i+} \\ &= \begin{pmatrix} \phi_1(\mathbf{p}_i) & 0 \\ 0 & \phi_2(\mathbf{p}_i) \end{pmatrix} P^{i+}. \end{aligned}$$

This is only possible if $\Psi(\mathbf{p}_i) = I$, but this is impossible, as before. This is our contradiction. Therefore, Ψ is not diagonalisable. \square

This concludes the proof of Theorem 0.2, and this paper.

REFERENCES

- [Ag185] Jim Agler, *Rational dilation on an annulus*, The Annals of Mathematics **121** (1985), no. 3, 537–563.
- [AHR08] Jim Agler, John Harland, and Benjamin J. Raphael, *Classical function theory, operator dilation theory, and machine computation on multiply-connected domains*, Memoirs of the American Mathematical Society **191** (2008), no. 892, viii+159.
- [AM02] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, American Mathematical Society, 2002.
- [Bar75] William H. Barker, *Plane domains with hyperelliptic double*, Ph.D. thesis, Stanford, 1975.
- [Bar77] ———, *Kernel functions on domains with hyperelliptic double*, Transactions of the American Mathematical Society **231** (1977), no. 2, 339–347.
- [DM05] Michael A. Dritschel and Scott McCullough, *The failure of rational dilation on a triply connected domain*, Journal of The American Mathematical Society **18** (2005), no. 4, 873–918.
- [DM07] ———, *Test functions, kernels, realizations and interpolation*, Operator Theory, Structured Matrices and Dilations: Tiberiu Constantinescu Memorial Volume, pp. 153–179, Theta Foundation, Bucharest, 2007.
- [Fay73] John D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, Springer-Verlag, 1973.
- [Fis83] Stephen D. Fisher, *Function theory on planar domains*, Pure and Applied Mathematics, Wiley-Interscience, 1983.

⁴ as this would mean it took the value 1 at least once on B_0, B_2, \dots, B_n , and at least twice on B_1 , so at least $n + 2$ times.

- [FK92] Hershel M. Farkas and Irwin Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics, Springer-Verlag, 1992.
- [Mum83] David Mumford, *Tata lectures on theta*, Progress in Mathematics, vol. I, Birkhäuser, 1983.
- [Mum84] ———, *Tata lectures on theta*, Progress in Mathematics, vol. II, Birkhäuser, 1984.
- [Neh52] Zeev Nehari, *Conformal mappings*, 1st ed., International series in pure and applied mathematics, McGraw-Hill, 1952.
- [Pau02] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, Cambridge, 2002.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE-UPON-TYNE, NEWCASTLE-UPON-TYNE, UNITED KINGDOM, NE1 7RU

E-mail address: james.pickering@ncl.ac.uk

URL: <http://www.jamespic.me.uk>