Nevanlinna-Pick Interpolation

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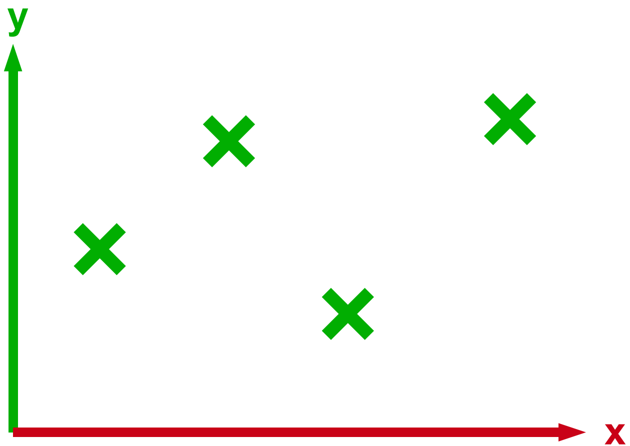
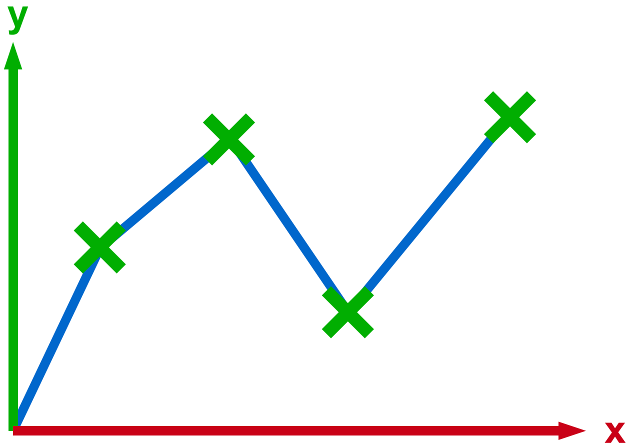
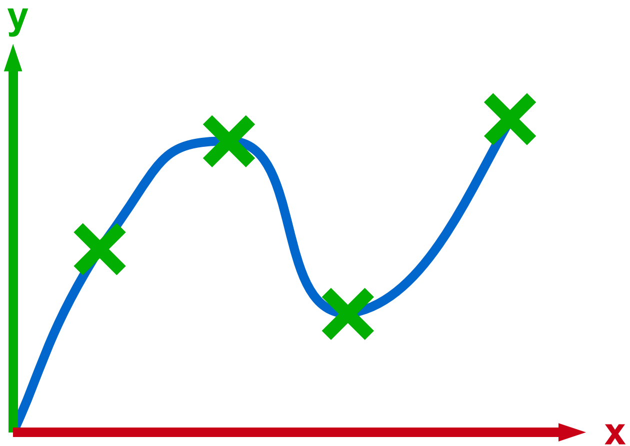
Interpolation



Interpolation



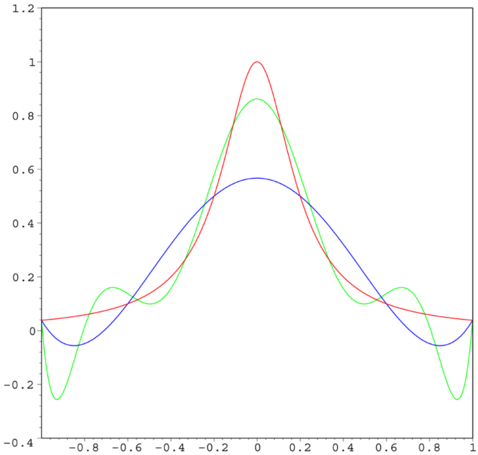
Interpolation



Nice Functions

The stereotypical “Nice Function” is a polynomial. If we have n points, we know we can choose a polynomial of order $n-1$ that passes through all of them. Naïvely, we might try interpolating this way.

The Problem



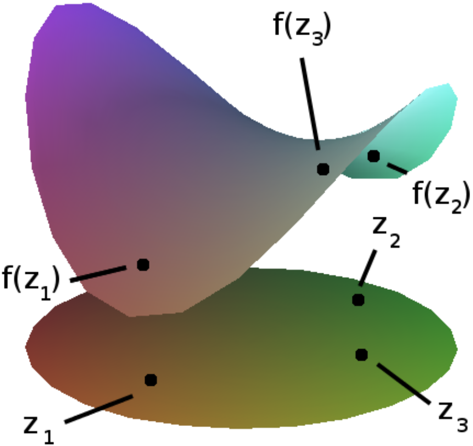
What went wrong?

The problem is *Runge's Phenomenon*. If we try to interpolate the points with a polynomial *of minimum degree*, as in Vandermonde or Lagrange interpolation, we can end up with "ringing" artifacts, which get bigger and bigger as we add more points.

Since there's no limit to how big these artifacts can become, one question we could ask is whether we can *still* come up with an interpolating function, if we *do* have some limit on how big they can become.

The Setup

- Polynomials can be problematic, so we use the next best thing – holomorphic functions.
- If we're going to use holomorphic functions, it's no extra hassle to let our points be complex, rather than just real.
- We want to put a limit on how big our interpolating function can be, but we know that holomorphic functions can't be bounded everywhere, so we work in the unit disc $\mathbb{D}$ instead. We'll require that $|f(z)| \leq 1$ for $z \in \mathbb{D}$ (we write this as $\|f\| \leq 1$).


f(z3)
f(z2)
z2
f(z1)
z1
z3

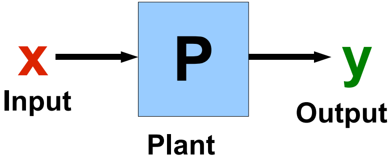
Motivation

Finding an interpolating polynomial is already a tough computational job; putting limits on how big our interpolating function can be makes the job even harder. If we're just looking to approximate a function for some numerical computation, there are easier ways to do this, so why bother?

An unlikely source of problems of this sort, is control engineering.

Some Engineering

Control engineering deals with “plants”. These can be anything that takes an input and gives an output: from a household thermostat, to a car’s fuel injector.



Assumptions

Before we can do maths, we need some assumptions about plants. We assume:

- Plants obey the laws of causality – future inputs can't affect past outputs.
- Plants are "stable". Usually, this means the output has less energy than the input.
- Plants are time invariant – it doesn't matter what time the plant was switched on.
- Plants are linear.

The z-transform

For convenience, we'll also work with discrete time, so we have a sequence of inputs $(x_0, x_1, x_2, \ldots)$ and a sequence of outputs $(y_0, y_1, y_2, \ldots)$. We can turn a sequence into a holomorphic function, by taking the terms as coefficients of a power series, so

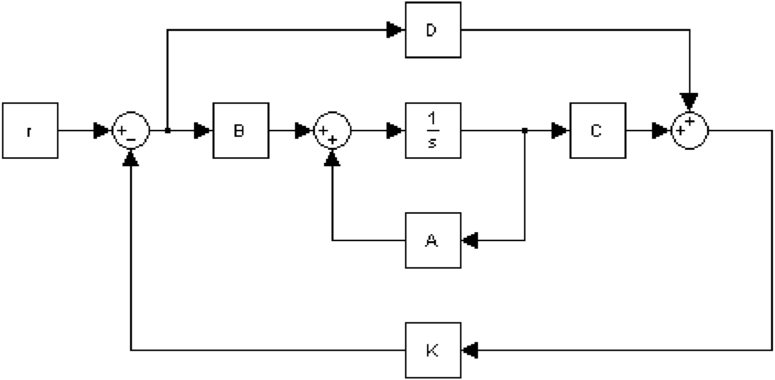
$$x \text{ goes to } \tilde{x} = \sum_{i=0}^{\infty} x_i z^i$$

When we do this, we find that for any plant P, there is a bounded, holomorphic function f_P such that

$$\tilde{y} = f_P \tilde{x}$$

In other words, the z-transform turns plants into **multiplication operators**, so we can work with plants by working with bounded, holomorphic functions.

A scary control diagram



Applications

In general, the systems control engineers study have a number of plants. Their job is usually to design a particular plant – a particular part of the system – so that the system as a whole will behave as required, or perform to a particular standard.

Since plants can be thought of as holomorphic functions, it's often possible to turn requirements for the system into requirements for the holomorphic function, which is where ideas like Nevanlinna-Pick interpolation come in.

A definition

We say an $n \times n$ matrix A is positive if

$$\langle A\mathbf{x},\ \mathbf{x}\rangle \geq 0$$

for all vectors $\mathbf{x} \in \mathbb{C}^n$. $\langle\ ,\ \rangle$ denotes the inner product, or dot product.

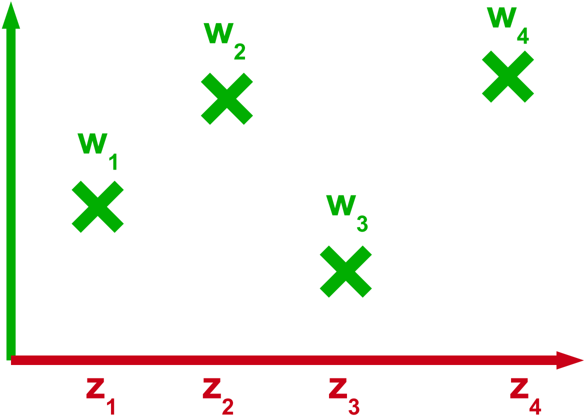
The Solution

Nevanlinna and Pick both discovered the following solution at roughly the same time (late 1910s):

There is a holomorphic function f, with $\|f\| \leq 1$, which takes $z_1, z_2, \ldots, z_n$ to $w_1, w_2, \ldots, w_n$ if and only if the matrix

$$A_{ij} := \left(\frac{1 - w_j \overline{w_i}}{1 - z_j \overline{z_i}} \right)$$

is positive.


w
2
w
4
w
1
w
3
z
1
z
2
z
3
z
4

A Technical Note

If we define

$$k(x,\ y) = \frac{1}{1 - y\overline{x}}$$

then $A_{ij} := (1 - w_j\overline{w_i})k(z_i,\ z_j)$. It turns out that k has some interesting properties, that tie in with this.

Reproducing Kernels

The set of “nice” holomorphic functions on the unit disc is a Hilbert space (called H^2) with inner product

$$\langle f(z),\ g(z)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dt$$

In fact, this space is a **reproducing kernel Hilbert space**, which means that there is a function $K_y(z)$ such that

$$f(y) = \langle f(z),\ K_y(z)\rangle$$

We can show that $K_y(z)$ is in fact $k(y,\ z)$, from before.

This is relevant, because bounded holomorphic functions are multiplication operators. In fact, the space they operate on is H^2.

It's possible to generalise the Nevanlinna-Pick theorem to other types of holomorphic function, and most of the generalisations work by turning functions into multiplication operators, and finding reproducing kernels for the spaces they operate on.

The Real World

Although we can use these techniques in control engineering (called H^∞ control theory), when we use H^∞ techniques to measure the performance of a system, the numbers can be misleading.

H^∞ techniques measure the **worst case** performance of a system. However, this can lead to systems whose average performance is less than it could be. H^∞ techniques are most useful when a system has to perform to a minimum standard.

A Real World Use

The End