Some Applications of The Theory of Test Function Realisations

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Abstract. We give two applications of the theory of test function realisations. These were developed in connection with interpolation problems. In the first, we use test function methods to construct a counterexample to the rational dilation conjecture. That is, we give a class of spaces and operators on them, such that the operators do not have rational boundary dilations. The second is a test function realisation for the algebra of functions on the disc with zero derivative at the origin. The methods used in this second application could also be used to find test function realisations for other algebras, or in interpolation problems.

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Contents

CHAPTER 1

Introduction

The purpose of this thesis is to illustrate some applications of test function realisations to problems in function theory and operator theory.

Test functions first appeared in connection with the Pick interpolation problem on the bidisc (see [**Agl90b**][1]) and subsequent work has expanded its use to other types of interpolation problem (see [**AM02**], [**DMM07**], [**MS04**], [**DM07**]).

The connections between interpolation and operator theory go back much further than this however; Sarason was the first to note (see [**Sar67**]), that the Nevanlinna-Pick and Carathéodory interpolation problems could both be seen as different facets of the same operator based problem. Numerous authors (far too many to hope to list) have expanded on these techniques, to solve related interpolation problems by operator theoretical means (for example [**Abr79**], [**McC96**], [**DPRS07**], [**BBT08**], [**MP02**], [**Rag08a**], [**Rag08b**], [**DP98**], [**Agl90a**]).

In this thesis, we will look at two problems related to test functions. In Chapter 3 we consider the first problem, the rational dilation problem, which concerns Hardy spaces $H^\infty(R)$ on planar domains $R \subseteq \mathbb{C}$. In essence, it conjectures that their contractive representations are completely contractive. Sz.-Nagy's dilation theorem (a proof of which can be found in [**Pau02**]) shows that the conjecture holds when $R = \mathbb{D}$, the unit disc. Agler showed (see [**Agl85**]) that the conjecture holds when R is an annulus. However, subsequent work (see [**AHR08**] and [**DM05**]) has shown that it does not hold if R is a triply connected domain (that is, R has two holes).

If we have a collection of test functions Ψ, then the set of functions they realise forms a normed algebra $H^\infty(\mathcal{K}_\Psi)$. Our first step will be to find a set of test functions Ψ so that $H^\infty(\mathcal{K}_\Psi) = H^\infty(R)$. We will see later that test functions can characterise contractive representations. The situation for completely contractive representations is somewhat more complex, and the

[1]It appears that a solution to the Nevanlinna-Pick interpolation problem on the bidisc was known as early as 1988 (there are references scattered throughout the literature, to an unpublished manuscript "Some interpolation theorems of Nevanlinna-Pick type" by Jim Agler) but [**Agl90b**] appears to be the first time it appeared in print.

theory is less complete, but we are able to show that a particular representation is not completely contractive, thus disproving the rational dilation conjecture for a particular class of three-or-more-holed domain R.

In Chapter 4 we consider the second problem. We are looking at the normed algebra

$$H_1^\infty = \{f \in H^\infty : f'(0) = 0\} .$$

This algebra has already been investigated in various papers (see for example [**DPRS07**], [**BBT08**], [**Rag08b**]). We sought to find a test function realisation for the algebra H_1^∞. To do this, first we find a representation similar to Herglotz' representation – a so called *Agler-Herglotz representation* (the realisations found in [**Agl90b**], [**DM07**], and indeed in Theorem 3.2.10, are based on this same idea. Herglotz representation 1 from [**AHR08**] is also similar), and then distill a realisation from it.

Before this though, we give some introductory background reading. In Chapter 2 we discuss test functions. We give definitions for terms we will need later, and recall some commonly used results in the field.

CHAPTER 2

Background on Test Function Realisations

Test function realisations first appeared in connection with the interpolation problem on the polydisc. The earliest work in this field was Agler's realisation of holomorphic functions on the polydisc, in [**Agl90b**] in 1990. Various other important techniques in this field were introduced in the late nineties, in [**BT98**] and [**AM99**].

It is hard to pin down precisely when the idea of an abstract theory of test function realisations first appeared. The idea was clearly known by 2002, as Agler discusses the abstract theory of test functions in [**AM02**]. However, authors were already using these kinds of techniques in novel settings by that point; it is unclear whether anyone had considered an abstract theory of test functions prior to this.

For our purposes, it will be most convenient to consider the treatment given in [**DM07**].

2.1. Definitions

We need to define what is meant by a *collection of test functions*. We generally use the notation $\psi|_F$ to denote the restriction of ψ to F. We also define a kernel, as a function $k : X \times X \to \mathbb{C}$; a kernel is positive if for any finite set of points $F = \{x_1, \ldots, x_n\}$, the matrix $\big(k(x_i, x_j)\big)_{i,j}$ is positive.

Definition 2.1.1. A collection Ψ of test functions is a collection of complex valued functions on a set X, satisfying the following conditions:

(1) For each $x \in X$,

$$\sup\{|\psi(x)| : \psi \in \Psi\} < 1 ,$$

(2) For each finite set F with n elements, the unital algebra generated by $\Psi|_F$ is n-dimensional

We also require that Ψ is a topological space. Condition 2 essentially says that the algebra generated by Ψ separates finite sets of points.

We do not need to assume that X is a topological space, as X inherits a topology from the test functions. To see this, define $C_b(\Psi)$ as the space of all bounded, continuous functions on Ψ. The evaluation mapping

$$E: X \to C_b(\Psi) \quad E(x)(\psi) = \psi(x)$$

is now an injective mapping[1], and defines a topology on X, if we assume it to be continuous.

The main reason we define test functions, is that a collection of test functions has an associated normed algebra, defined in the following way.

DEFINITION 2.1.2. If Ψ is a collection of test functions, then we write

$$\mathcal{K}_\Psi := \left\{k: X \times X \to \mathbb{C} \mid k \geq 0, \left(1 - \psi(x)\overline{\psi(y)}\right) k(x, y) \geq 0 \, \forall \psi \in \Psi\right\}.$$

Dually, if we have a collection $\mathcal{K}$ of positive kernels over X, we can define a normed algebra of functions on X. The quickest way to define the normed space $H^\infty(\mathcal{K})$ is to define its unit ball:

$$\mathcal{B}H^\infty(\mathcal{K}) := \left\{\varphi: X \to \mathbb{C} \mid \left(1 - \varphi(x)\overline{\varphi(y)}\right) k(x, y) \text{ pos.} \forall k \in \mathcal{K}\right\}.$$

Addition and multiplication are defined pointwise.

2.2. The Realisation Theorem

The key result in this area is that there are numerous, equivalent, ways of describing $H^\infty(\mathcal{K}_\Psi)$.

THEOREM 2.2.1 ([**DM07**]). *If Ψ is a collection of test functions, the following are equivalent:*

(1) $\varphi \in H^\infty(\mathcal{K}_\Psi)$ *and* $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1$.

(2) (a) *For each finite set $F \subseteq X$, there exists a positive kernel*[2] $\Gamma : F \times F \to C_b(\Psi)^*$, *such that for all* $x, y \in F$,

$$1 - \varphi(x)\overline{\varphi(y)} = \Gamma(x, y)\left(1 - E(x)E(y)^*\right).$$

(b) *There exists a positive kernel* $\Gamma : X \times X \to C_b(\Psi)^*$ *such that for all* $x, y \in X$,

$$1 - \varphi(x)\overline{\varphi(y)} = \Gamma(x, y)\left(1 - E(x)E(y)^*\right).$$

(3) *There exists a Hilbert space $\mathcal{E}$, a unitary operator*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{E} \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} \mathcal{E} \\ \oplus \\ \mathbb{C} \end{matrix},$$

and a unital $*$-representation

$$\rho : C_b(\Psi) \to B(\mathcal{E}),$$

[1] the mapping is injective by part 2 of the definition

[2] This kernel is positive, in the sense that $\Gamma(\cdot, \cdot)(L)$ is a positive kernel at each $L \in C_b(\Psi)$

such that

$$\varphi(x) = D + CZ(x)\,(I - AZ(x))^{-1}\,B\,,$$

where $Z(x) = \rho(E(x))$.

(4) (a) *For every representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *such that* $\|\pi(\psi)\| < 1$ *for all* $\psi \in \Psi$*, we have that* $\|\pi(\varphi)\| \leq 1$.
(b) *For every weakly continuous*[3] *representation* π *of* $H^\infty(\mathcal{K}_\Psi)$ *such that* $\|\pi(\psi)\| \leq 1$ *for all* $\psi \in \Psi$*, we have that* $\|\pi(\varphi)\| \leq 1$.

The various equivalent formulations given in this theorem first appeared in various forms in various places.

Functions that satisfy condition 1 form the *Schur-Agler class*. Condition 2 first appeared in [**Agl90b**], where Agler gave sets of test functions Ψ_1 and Ψ_2 (the notation here is more modern than the notation used at the time) such that $H^\infty(\mathbb{D})$ is isometrically isomorphic to $H^\infty(\Psi_1)$ and $H^\infty(\mathbb{D}^2)$ is isometrically isomorphic[4] to $H^\infty(\Psi_2)$ (some authors say that the kernels Γ model φ, although we do not use that piece of terminology here). This also appears to be the first time that condition 3 was used in this context.

Condition 3 says that φ is the *transfer function of a unitary colligation*. This particular terminology is motivated by control theory, as is the paper [**BT98**], which introduced many of the techniques and notations used today.

Condition 4 is the generalised von Neumann inequality, and had been shown to hold in the case of $H^\infty(\mathbb{D})$ by von Neumann, and in the case of $H^\infty(\mathbb{D}^2)$ by Andô. Proofs of both of these classical results can be found in [**Pau02**].

A definition of $H^\infty(\mathcal{K}_\Psi)$ similar to this one first appeared in print in [**AM02**], although it is not clear how widely this idea was understood before that time.

We do not reproduce the proof of Theorem 2.2.1 here. However, many of the techniques used in [**DM07**] to prove this result can be found in Section 3.5. The results in Section 3.5 amount to a partial proof of a matrix-valued generalisation of Theorem 2.2.1, although at the time of writing, no complete matrix-valued generalisation of Theorem 2.2.1 is known.

A consequence of Theorem 2.2.1 is the following interpolation theorem, previously given in [**DM07**].

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[3]When we say that a representation $\pi : H^\infty(\mathcal{K}_\Psi) \to B(\mathcal{J})$ is weakly continuous, we mean that $H^\infty(\mathcal{K}_\Psi)$ has the topology of pointwise convergence, and $B(\mathcal{J})$ has the weak operator topology.

[4]Agler actually gave sets of test Ψ_n for $H^\infty(\mathbb{D}^n)$ for all $n \in \mathbb{N}$, but the resulting isomorphisms are only isometric when $n = 1$ or 2

THEOREM 2.2.2. *If we have a set of test functions Ψ on a space X, a finite subset $F \subset X$, and a function $\zeta : F \to \mathbb{D}$, then the following are equivalent:*

(1) *There exists a $\varphi \in H^\infty(\mathcal{K}_\Psi)$ with $\|\varphi\| \leq 1$ and $\varphi|_F = \zeta$.*
(2) *For each $k \in \mathcal{K}_\Psi$, the kernel*

$$F \times F \ni (x, y) \to \left(1 - \zeta(x)\overline{\zeta(y)}\right) k(x, y)$$

is positive.
(3) *There exists a positive kernel $\Gamma : F \times F \to C_b(\Psi)^*$ so that for all $x, y \in F$*

$$1 - \zeta(x)\overline{\zeta(y)} = \Gamma(x, y)\left(1 - E(x)E(y)^*\right) .$$

Further, if Ψ is compact, there is a bounded positive measure μ on Ψ so that

$$\varphi(x) = D + CE(x)\left(I - AE(x)\right)^{-1} B$$

for some unitary

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathbb{C}^n \otimes L^2(\mu) \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} \mathbb{C}^n \otimes L^2(\mu) \\ \oplus \\ \mathbb{C} \end{matrix} .$$

This can be seen as analogous to the Nevanlinna-Pick interpolation theorem (a modern treatment of which can be found in [**AM02**]), which is as follows:

THEOREM 2.2.3. *If we have a finite subset $F \subset \mathbb{D}$, and a function $\zeta : F \to \mathbb{D}$, then the following are equivalent:*

(1) *There exists a $\varphi \in H^\infty(\mathbb{D})$ with $\|\varphi\| \leq 1$ and $\varphi|_F = \zeta$.*
(2) *The kernel*

$$F \times F \ni (x, y) \to \frac{1 - \zeta(x)\overline{\zeta(y)}}{1 - x\overline{y}}$$

is positive.

CHAPTER 3

Rational Dilation on Multiply Connected Domains

This chapter is based on material previously published by the author in [**Pic08a**]. Readers should be familiar with the theory of Riemann surfaces. See [**FK92**] for an introduction

3.0.1. Definitions for this Chapter. Let X be a compact, path connected subset of $\mathbb{C}$, with interior R, and *analytic boundary* B composed of $n+1$ disjoint curves, $B_0, \ldots, B_n$, where $n \geq 2$. By analytic boundary, we mean that for each boundary curve B_i there is some biholomorphic map ϕ_i on a neighbourhood U_i of X which maps B_i to the unit circle $\mathbb{T}$. By convention B_0 is the outer boundary. We write $\Pi = B_0 \times \cdots \times B_n$.

We say a Riemann surface Y is *hyperelliptic* if there is a meromorphic function with exactly two poles on Y. We say R is symmetric if there exists some anticonformal involution ϖ on R with $2n+2$ fixed points on B. We say a domain in $\mathbb{C} \cup \{\infty\}$ (that is, the Riemann sphere S^2) is a *real slit domain* if its complement is a finite union of closed intervals in $\mathbb{R} \cup \{\infty\}$.

We define $\mathcal{R}(X) \subseteq C(X)$ as the space of all rational functions that are continuous on X. The definitions of contractivity and complete contractivity are the usual definitions, and can be found in [**Pau02**].

3.0.2. The Problem. In this chapter, we will give a partial resolution to the *rational dilation conjecture*, which is as follows.

Conjecture 3.0.4. *If $X \subseteq \mathbb{C}$ is a compact domain, $T \in \mathcal{B}(H)$ is a Hilbert space operator with $\sigma(T) \subseteq X$ and $\left\|f(T)\right\| \leq 1$ for all $f \in \mathcal{R}(X)$ with $\left\|f\right\|_{C(X)} \leq 1$, then there is some normal operator $N \in \mathcal{B}(K)$, $K \supseteq H$, such that $\sigma(N) \subseteq B$ $(= \partial X)$, and $f(T) = P_H N|_H$.*

A classical result of Sz.-Nagy shows that the rational dilation conjecture holds if X is the unit disc. A generalisation by Berger, Foias and Lebow shows this holds for any simply connected domain (see [**Pau02**]). A result by Agler (see [**Agl85**]) shows that rational dilation also holds if X has one hole – such as in an annulus. However, subsequent work has shown that rational dilation fails on every two-holed domain with analytic boundary (see [**DM05**], and [**AHR08**]).

We will prove the following, which by a result of Arveson (see [**Pau02**, Cor. 7.8]), is equivalent to showing that the rational dilation conjecture does not hold on any symmetric, two-or-more-holed domain.

Theorem 3.0.5. *If X is a symmetric domain in $\mathbb{C}$, with $2 \leq n < \infty$ holes, there is an operator $T \in \mathcal{B}(H)$, for some Hilbert space H, such that the homomorphism $\pi : \mathcal{R}(X) \to \mathcal{B}(H)$ with $\pi(p/q) = p(T) \cdot q(T)^{-1}$ is contractive, but not completely contractive.*

Proof Outline. First, we let C define the cone generated by

$$\left\{H(z)\left[1 - \psi(z)\overline{\psi(w)}\right]H(w)^* : \psi \in \mathcal{B}H^\infty(X), H \in M_2\left(H^\infty(X)\right)\right\},$$

where $\mathcal{B}H^\infty(X)$ is the unit ball of the space of functions analytic in a neighbourhood of X, under the supremum norm, and $M_2\left(H^\infty(X)\right)$ is the space of 2×2 matrix valued functions analytic in a neighbourhood of X. For $F \in M_2\left(H^\infty(X)\right)$, we set

$$\rho_F = \sup\left\{\rho > 0 : I - \rho^2 F(z)F(w)^* \in C\right\}.$$

We show that there exists a function F which is unitary valued on B (we say F is *inner*), but such that $\rho_F < 1$. We show that such a function generates a counter-example of the type needed. To show that such a function exists, we show that if F is inner, $\rho_F = 1$ ($\|F\| = 1$ by the max modulus principle, so $\rho_F \leq 1$), and the zeroes of F are "well behaved", then F can be diagonalised. We go on to show that there is a non-diagonalisable inner function F, with well behaved zeroes, which must therefore have $\rho_F < 1$, so must be a counter-example. ◻

By [**BC67**], we know that any two-holed domain is symmetric, so the result of [**DM05**] can be seen as a consequence of Theorem 3.0.5.

In [**DM05**] and [**AHR08**], the authors work with domains whose boundaries are circles, with centres on the real line. Such domains are clearly symmetric. The result of [**BC67**] shows that any two-holed domain is conformally equivalent to one of this form. More generally, Koebe's Theorem (see [**Haz02**] or [**GL69**]) shows that any multiply-connected domain R can be conformally mapped to a circular domain, although the circles need not line up along the real line.

3.1. Symmetries

Details of the ideas discussed below can be found in [**Bar75**].

Theorem 3.1.1. *Let $R \subseteq \mathbb{C}$ have $n+1$ analytic boundary curves, $B_0, \ldots, B_n \subseteq B$, with $n \geq 2$, and let Y be its Schottky double. The following are equivalent:*

(1) Y is hyperelliptic;
(2) R is symmetric;
(3) R is conformally equivalent to a real slit domain Ξ.

The proof can be found in [**Bar75**], but we will briefly discuss the constructions involved. We know from [**FK92**, III.7.9] that Y is hyperelliptic if and only if there is a conformal involution $\iota : Y \to Y$ with $2n + 2$ fixed points. We find that ι is given by

$$\iota(x) = \begin{cases} J \circ \varpi(x) & x \in R \\ \varpi(x) & x \in B \\ \varpi \circ J(x) & x \in J(R) \end{cases},$$

where J is the "mirror" function on Y, that maps the front version of X to the back version, and vice versa.

Also, if $\varsigma : \Xi \to R$ is the conformal mapping from part 3, we have that $\varpi(\varsigma(\xi)) = \varsigma\left(\overline{\xi}\right)$.

Definition 3.1.2. We define the *fixed point set* of our symmetric domain R as

$$\mathbb{X} := \{x \in R : x = \varpi(x)\} .$$

Remark 3.1.3. In view of Theorem 3.1.1 on the preceding page, it makes sense to relabel the components of B. We can see that $\mathbb{X}$ must be the image of $\mathbb{R} \cap \Xi$ under ς, so must consist of a finite collection of paths running between fixed points of B. We choose one of the two fixed points of B_0, and call it p_0^-. We follow $\mathbb{X}$ from p_0^- to another B_i which we relabel B_1; we call the fixed point we landed at p_1^+. Label the other fixed point in B_1 as p_1^-, and repeat, until we reach p_0^+. The section of $\mathbb{X}$ from p_i^- to p_{i+1}^+, we call $\mathbb{X}_i$. See Figure 3.1.1.

Proposition 3.1.4. *If a meromorphic function f on the above-mentioned hyperelliptic surface Y has n or fewer poles, and all of these poles lie in $R \cup B$, then all of these poles must lie on B.*

Proof. Suppose f has n or fewer poles. Then $f \circ \iota$ also has n or fewer poles, so $f - f \circ \iota$ has $2n$ or fewer poles. However, if x is a fixed point of ι, $f(x) - f \circ \iota(x) = 0$, and since ι has $2n + 2$ fixed points, $f - f \circ \iota$ has at least $2n + 2$ zeroes. This is only possible if $f - f \circ \iota \equiv 0$, so if x is a pole of f, then $\iota(x)$ is a pole of f, which is a contradiction unless $x \in B$. □

3.2. Inner Functions

In this section, we describe a class of measures on B, corresponding to period free harmonic functions on R. This is done by a shorter method in

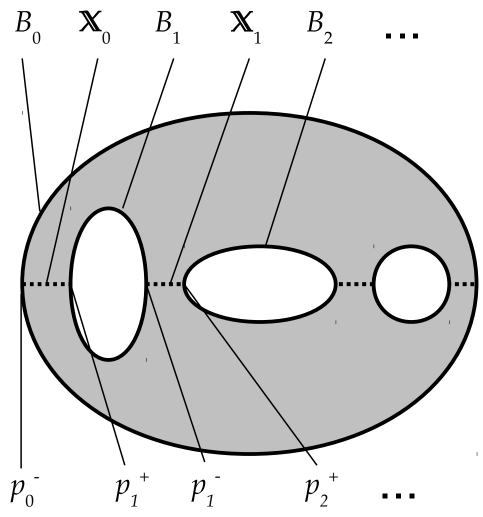

FIGURE 3.1.1. The structure of X

[**AHR08**] (we will in fact use a variant of their method to tackle a different problem, in Section 4.3.2), but the method described here is constructive. The approach taken here has more in common with the approach taken in [**DM05**]. Those unfamiliar with the function theory used can find an excellent introduction in [**Fis83**].

Results in this section often require us to choose a fixed point $b \in R$. Usually, b will be determined by the particular application, but in this section we make no requirements on the choice of b. We can choose, and fix, any point, and use it as our b. Similarly, none of the results in this section assume that R is symmetric.

3.2.1. Harmonic and Analytic Functions. If ω_b is harmonic measure at b, and s is arc length measure, by an argument like the one in [**DM05**], we can find a Poisson kernel $\mathbb{P} : R \times B \to \mathbb{R}$ such that for h harmonic on R and continuous on B,

$$h(w) = \int_B h(z)\mathbb{P}(w, z)ds(z) .$$

Equivalently, $\mathbb{P}$ is given by the Radon-Nikodým derivative

$$\mathbb{P}(w, \cdot) = \frac{d\omega_w}{ds} .$$

We know that $\mathbb{P}$ is harmonic in R at each point in B, and that for any positive h harmonic on R, and continuous on X there exists some positive measure μ on B such that

$$h(w) = \int_B \mathbb{P}(w, z)d\mu(z) .$$

Conversely, given a positive measure μ on B, this formula defines a positive harmonic function.

We let h_j denote the solution to the Dirichlet problem which is 1 on B_j and 0 on B_i, where $i \neq j$. We can see that this corresponds to the arc length measure on B_j.

We define $Q_j : B \to \mathbb{R}$ as the outward normal derivative of h_j, and define the *periods* of h by

$$P_j(h) := \int_B Q_j d\mu .$$

It should be clear that h is the real part of an analytic function if and only if $P_j(h) = 0$ for $j = 0, 1, \dots, n$.

Lemma 3.2.1. *The functions Q_j have no zeroes on B. Moreover, $Q_j > 0$ on B_j and $Q_j < 0$ on B_l for $l \neq j$.*

Proof. As X has analytic boundary, we can assume without loss of generality that $B_0 = \mathbb{T}$. We know that h_j takes its minimum and maximum on its boundary. Since h_j equals one on B_j, and zero on B_l if $l \neq j$, these must be its maximum and minimum respectively, so h_j is non-decreasing towards B_j, and non-increasing towards B_l, so $Q_j \geq 0$ on B_j and $Q_j \leq 0$ on B_l.

We can see by the above argument that we only need show that $Q_j \neq 0$. We let R' be the reflection of R about B_0 (which we are assuming is the unit circle). We can extend h_j to a harmonic function on $X \cup R'$ by setting

$$h_j(z) = -h_j(1/\bar{z})$$

on R'.

If Q_j had infinitely many zeroes on B_0, then Q_j would be identically zero, so we suppose Q_j has finitely many zeroes on B_0.

Suppose Q_j has a zero z, and a small, simply connected neighbourhood $N(z)$. By choosing $N(z)$ small enough, we can ensure that $N(z)$ contains no other zeroes. Clearly, h_j forms the real part of some holomorphic function f on $N(z)$. We know that $\partial h_j / \partial n = Q_j = 0$, and because h_j is constant on B_0, we know that the tangential derivative of h_j, $\partial h_j / \partial t$, is also zero, so f has derivative zero at z, so f has a ramification of order at least two at z. We also know that f maps everything outside the unit disc to the left half plane, and everything inside the unit disc to the right half plane, but clearly this is impossible, so Q_j cannot have a zero.

A similar argument holds for $B_1, \dots, B_n$. □

Corollary 3.2.2. *If h is a non-zero positive harmonic function on R which is the real part of an analytic function, and h is represented in terms of a positive measure μ, then $\mu(B_j) > 0$ for each j.*

Proof. If $\mu(B_j) = 0$, then as $Q_j < 0$ on $B\backslash B_j$, $P_j(h) < 0$, a contradiction. Thus, $\mu(B_j) > 0$. □

3.2.2. Some Matrix Algebra. We wish to show that at each $p \in \Pi$, the vector

$$V^n = \det\begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ Q_1(p_0) & Q_1(p_1) & \cdots & Q_1(p_n) \\ Q_2(p_0) & Q_2(p_1) & \cdots & Q_2(p_n) \\ \vdots & \vdots & \ddots & \vdots \\ Q_n(p_0) & Q_n(p_1) & \cdots & Q_n(p_n) \end{pmatrix}$$

has only positive coordinates. It helps to note that in three dimensions

$$\mathbf{x} \times \mathbf{y} = \det\begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix}.$$

It will also be helpful to write

$$V^n = \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_n \\ - & + & - & - & \cdots & - \\ - & - & + & - & \cdots & - \\ - & - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & - & \cdots & + \end{vmatrix},$$

noting that $Q_j(p_j) > 0$, and $Q_i(p_j) < 0$ for $i \neq j$. From here on, positive and negative quantities will simply be denoted by (+) and (−), respectively.

Lemma 3.2.3. *All sub-matrices of V^n of the form*

$$\begin{pmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \cdots & + \end{pmatrix}$$

have positive determinant.

Proof. We can assume, without loss of generality, that such matrices are of the form

$$\begin{pmatrix} Q_1(p_1) & Q_1(p_2) & \cdots & Q_1(p_k) \\ Q_2(p_1) & Q_2(p_2) & \cdots & Q_2(p_k) \\ \vdots & \vdots & \ddots & \vdots \\ Q_k(p_1) & Q_k(p_2) & \cdots & Q_k(p_k) \end{pmatrix} := A^T$$

by a simple relabelling of boundary curves. We note that

$$\sum_{j=0}^{n} h_j \equiv 1 \,,$$

so in particular

$$\sum_{j=0}^{n} Q_j(x) = 0$$

for all $x \in B$. So, if $1 \le i \le k$, then

$$\sum_{j=1}^{k} Q_j(p_i) = -\left(Q_0(p_i) + \sum_{j=k+1}^{n} Q_j(p_i) \right) > 0 \,.$$

We now apply Gershgorin's circle theorem. Since $A_{ij} = Q_j(p_i)$, the eigenvalues of A are in the set

$$S := \bigcup_{i=1}^{N} D\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} A_{ij}, A_{ii} \right) := \bigcup_{i=1}^{N} S_i \,,$$

where $D(\epsilon, x) \subseteq \mathbb{C}$ is the ball centred at x of radius ϵ. Now, if $\lambda \in S_i$, then $|\lambda - A_{ii}| < \sum_{j \neq i} A_{ij}$, so in particular

$$\Re(\lambda) > A_{ii} - \sum_{j \neq i} \left| A_{ij} \right| = A_{ii} + \sum_{j \neq i} A_{ij} = \sum_{j=1}^{n} A_{ij} > 0 \,.$$

Now, all terms in the matrix A are real, so if λ is an eigenvalue of A, then either $\lambda > 0$, or $\bar{\lambda}$ is also an eigenvalue. We know that the determinant of a matrix is given by the product of its eigenvalues, counting multiplicity. Therefore, the determinant of A is a product of positive reals, and terms of the form $\lambda\bar{\lambda} = |\lambda|^2$, which are also positive and real, so $\det(A)$ is positive, so $\det\left(A^T\right)$ is positive. □

Lemma 3.2.4. *V^n has only positive coefficients.*

Proof. We define

$$d_i^n = \left|\begin{array}{c|c} \overbrace{\begin{matrix} - & + & - & \cdots & - \\ - & - & + & \ddots & \vdots \\ - & - & - & \ddots & - \\ \vdots & \vdots & \ddots & \ddots & + \\ - & \cdots & - & - & - \end{matrix}}^{i\times i} & - \\ \hline - & \begin{matrix} + & - & - & \cdots & - \\ - & + & - & \ddots & - \\ - & - & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & - \\ - & - & \cdots & - & + \end{matrix} \end{array}\right| .$$

For our purposes, all that matters is the signs of the elements of this matrix, and that Lemma 3.2.3 on page 15 holds. Cyclically permuting the first i rows gives

$$d_i^n = (-1)^{i-1} \begin{vmatrix} - & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & \vdots \\ - & \vdots & \ddots & \ddots & - \\ - & - & \cdots & - & + \end{vmatrix} = (-1)^{i-1} d_1^n .$$

We can see that

$$\begin{aligned} V^n = & \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_n \\ - & + & - & - & \cdots & - \\ - & - & + & - & \cdots & - \\ - & - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & - & \cdots & + \end{vmatrix} \\ = & (+)\mathbf{e}_0 + \sum_{i=1}^{n} (-1)^i d_i^n \mathbf{e}_i \end{aligned}$$

and

$$\begin{aligned} d_1^n = & (-(+)) - (-d_1^{n-1}) + (-d_2^{n-1}) - \cdots + (-1)^{n-1}(-d_{n-1}^{n-1}) \\ = & (-) + \sum_{j=1}^{n-1} (-1)^j (d_j^{n-1}) . \end{aligned}$$

We now proceed by induction. We first consider the case where $k = 1$. We can see that

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 \\ - & + \end{vmatrix} = (+)\mathbf{e}_0 - (-)\mathbf{e}_1 = (+)\mathbf{e}_0 + (+)\mathbf{e}_1\,,$$

so the lemma holds for $k = 1$. Now suppose that the lemma holds for $k-1$, and consider V^k. The $\mathbf{e}_0$ coordinate is positive, by Lemma 3.2.3 on page 15. The $\mathbf{e}_i$ coordinate is given by

$$\begin{aligned}(-1)^i d_i^k = (-1)^i(-1)^{i-1}d_1^k = (-)\left((-) + \sum_{j=1}^{k-1}(-1)^{j+1}(d_j^{k-1})\right)\\ = (+) + \sum_{j=1}^{k-1}\underbrace{(-1)^j(d_j^{k-1})}_{\mathbf{e}_j \text{ term of } V^{k-1}} = (+)\,,\end{aligned}$$

so the lemma holds for k, and so holds for all $k \in \mathbb{N}$. □

Corollary 3.2.5. *For each $p \in \Pi$, the kernel of*

$$M(p) = \begin{pmatrix} Q_1(p_0) & Q_1(p_1) & Q_1(p_2) & \cdots & Q_1(p_n) \\ Q_2(p_0) & Q_2(p_1) & Q_2(p_2) & \cdots & Q_2(p_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_n(p_0) & Q_n(p_1) & Q_n(p_2) & \cdots & Q_n(p_n) \end{pmatrix}$$

is one dimensional and spanned by a vector with strictly positive entries. Further, we can define a continuous function $\kappa : \Pi \to \mathbb{R}^{n+1}$ such that $\kappa(p)$ is entry-wise positive, and $\kappa(p)$ is in the kernel of $M(p)$.

Proof. We can see that $M(p)$ is always rank n, as the right hand $n \times n$ sub-matrix is invertible, by Lemma 3.2.3, so its kernel is everywhere rank one. If at each $p \in \Pi$ we take the V^n defined earlier, and define this as $\kappa(p)$, it is clear that this is entry-wise positive, orthogonal to the span of the row vectors (so in the kernel of $M(p)$), and has entries that sum to one, from the definitions and the above proved theorems. □

3.2.3. Canonical Analytic Functions. For $p \in \Pi$ we define

$$k_p = \sum_{j=0}^{n} \kappa_j(p)\mathbb{P}(\cdot, p_j)\,,$$

where κ is as in Corollary 3.2.5. Define $\tau : \Pi \to \mathbb{R}^{n+1}$ by $\tau(p) = \kappa(p)/k_p(b)$. We then define

$$h_p = \sum_{j=0}^{n} \tau_j(p)\mathbb{P}(\cdot, p_j)\,.$$

It is clear that this corresponds to the measure

$$\mu = \sum_{j=0}^{n} \tau_j(p)\delta_{p_j}$$

on B. We can see that h_p, thus defined, is a positive harmonic function, with $h_p(b) = 1$. We can also see that its periods are zero, as

$$P_j(h_p) = \int_B Q_j d\mu = \int_B Q_j \sum_{i=0}^{n} \tau_i(p)\delta_{p_i} = \sum_{i=0}^{n} \tau_i(p) \int_B Q_j\delta_{p_i} = \sum \tau_i(p)Q_j(p_i) = 0\,, \tag{3.2.1}$$

as $\tau(p)$ is in the kernel of $M(p)$, and (3.2.1) is just the j-th coordinate of $M(p)\tau(p)$. The function h_p is therefore the real part of an analytic function f_p on R. We require that $f_p(b) = 1$.

We define $\mathcal{H}(R)$ as the space of holomorphic functions on R, with the compact open topology. This is locally convex, metrisable, and has the Heine-Borel property, that is, closed bounded subsets of $\mathcal{H}(R)$ are compact. We then define

$$\mathbb{K} = \left\{ f \in \mathcal{H}(R) : \ f(b) = 1,\ f + \bar{f} > 0 \right\}\,.$$

Lemma 3.2.6. *The set $\mathbb{K}$ is compact.*

Proof. $\mathbb{K}$ is clearly closed, so it suffices to show that $\mathbb{K}$ is bounded. The case where R is the unit disc is proved in **[DM05]**, and we use this result without proof.

Since the $B_0, \ldots, B_n$ are disjoint, closed sets, and R is T_4, we can find disjoint open sets $U_0, \ldots, U_n$ containing each. By a simple topological argument we can show that there exists some $E > 0$ such that

$$O_i(E) := \{z \in \mathbb{C} : \ d(z, B_i) < E\} \subseteq U_i\,.$$

It is clear that R is covered by the family of connected compact sets

$$\{K_\epsilon\} := \left\{ R \backslash \left(\bigcup_i O_i(\epsilon) \right) : \ 0 < \epsilon < E \right\}\,,$$

so it is sufficient to work with just these compact sets.

We choose a sequence of disjoint, simple paths $v_0, \ldots, v_n$ through X such that v_i goes from B_i to B_{i+1}, and v_0 passes through b (note that when X is a symmetric domain, $v_i = \mathbb{X}_i$ satisfies this). It is clear that the union of these paths cuts X into two disjoint, simply connected sets U and V. It is also possible to show that we can choose a $\delta > 0$ such that adding

$$W := \{z \in R : d(z, v_i) \leq \delta \text{ for some } i\}$$

to either of these sets preserves simple connectivity. We can see that $K_\epsilon^+ := K_\epsilon \cap (U \cup W)$ and $K_\epsilon^- := K_\epsilon \cap (V \cup W)$ are simply connected compact sets containing b, whose union is K_ϵ. By the Riemann mapping theorem, we can canonically map $K_\epsilon^\pm$ to the unit disc, in a way that takes b to zero, so by the result of [**DM05**] mentioned earlier, we have a constant $M_\epsilon^\pm$, such that f analytic on R with $f(b) = 1$ implies for all $z \in K_\epsilon^\pm$, $|f(z)| \leq M_\epsilon^\pm$. □

LEMMA 3.2.7. *The extreme points of* $\mathbb{K}$ *are precisely* $\{f_p : p \in \Pi\}$.

PROOF. Clearly, each f_p is an extreme point of $\mathbb{K}$, so we prove the converse – if $f \neq f_p$, then f is not an extreme point of $\mathbb{K}$.

If $f \in \mathbb{K}$, then the real part of f is a positive harmonic function h with $h(b) = 1$. We therefore know that there is some positive measure μ on B such that

$$h(w) = \int_B \mathbb{P}(w, z) d\mu(z) .$$

As f is holomorphic, by Corollary 3.2.2 on page 14, μ must support at least one point on each B_i. If $f \neq f_p$, then μ must support more than one point on some B_i.

Now, a note. We know f is holomorphic if $P_j(h) = 0$ for $j = 0, \ldots, n$. However, we know that $\sum_{j=0}^n Q_j = 0$, so $\sum_{j=0}^n P_j(h) = 0$, so if we show that all but one of the $P_j(h)$ are zero, we have shown that they are all zero, so f is holomorphic.

With that in mind, suppose that μ supports more than one point on B_0. We do not lose any generality by doing this, as relabelling the boundary curves does not matter in the proof below, so we can safely relabel any given boundary curve B_0. We divide B_0 into two parts, A_1 and A_2, in such a way that μ is non-zero on both.

Now, let

$$a_{jl} = \int_{A_l} Q_j d\mu , \qquad l = 1, 2 ,$$

and

$$k_{jm} = \int_{B_m} Q_j d\mu , \qquad m = 1, \ldots, n ,$$

Since h is the real part of an analytic function,

$$0 = \int_B Q_j d\mu ,$$

so

$$\sum_{m=1}^n k_{jm} + a_{j1} + a_{j2} = 0 .$$

Since $Q_j < 0$ on B_i for $i \neq j$, for any $M \subseteq \{1, \ldots, n\}$ containing j,

$$\sum_{m \in M} k_{jm} = -\left(a_{j1} + a_{j2} + \sum_{m \notin M} k_{jm}\right) > 0 \,.$$

We can now apply the Gershgorin circles trick from the proof of Lemma 3.2.3 on page 15, to see that all sub-matrices of $K := (k_{jm})$ of the form

$$\begin{pmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \cdots & + \end{pmatrix}$$

have positive determinant (including K, which must therefore be invertible). We also note that the proof of Lemma 3.2.4 on page 16 only used this fact and the signs of the elements of matrices.

We consider the adjugate matrix C of K, which is defined by

$$c_{jm} = (-1)^{j+m} \left| \left(k_{\alpha\beta}\right)_{\substack{\alpha \neq j \\ \beta \neq m}} \right|$$

and has the property that $\det(K)^{-1} C^T = K^{-1}$. If we can show that all the c_{jm} are positive, then we will have that all the entries of K^{-1} are positive.

Now, if $j = m$, then

$$c_{jm} = \cancel{(-1)^{j+j}} \begin{vmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \cdots & + \end{vmatrix} = (+) \,.$$

If $m > j$ then c_{jm} is given by

$$(-1)^{j+m}\left|\begin{array}{c|c|c} \begin{matrix}+&-&\cdots&-\\ -&+&\cdots&-\\ \vdots&\vdots&\ddots&\vdots\\ -&-&\cdots&+\end{matrix} & - & - \\ \hline - & \overbrace{\begin{matrix}-&+&-&\cdots&-\\ -&-&+&\ddots&\vdots\\ -&-&-&\ddots&-\\ \vdots&\vdots&\ddots&\ddots&+\\ -&\cdots&-&-&-\end{matrix}}^{(m-j)\times(m-j)\text{ block}} & - \\ \hline - & - & \begin{matrix}+&-&\cdots&-\\ -&+&\cdots&-\\ \vdots&\vdots&\ddots&\vdots\\ -&-&\cdots&+\end{matrix}\end{array}\right| . \tag{3.2.2}$$

By cyclically permuting the $m-j$ rows in the middle we get

$$(-1)^{\cancel{j-m}-1}\cancel{(-1)^{j+m}}\left|\begin{array}{cc|c|c} & & \text{col } j & \\ & \begin{matrix}+&-&\cdots&-\\ -&+&\cdots&-\\ \vdots&\vdots&\ddots&\vdots\\ -&-&\cdots&+\end{matrix} & - & - \\ \hline \text{row } j & - & - & - \\ \hline & - & - & \begin{matrix}+&-&\cdots&-\\ -&+&\cdots&-\\ \vdots&\vdots&\ddots&\vdots\\ -&-&\cdots&+\end{matrix}\end{array}\right| ,$$

and by cyclically permuting the first j rows, and the first j columns we get

$$(-1)\cancel{(-1)^{j-1}}\cancel{(-1)^{j-1}}d_1^{n-1} ,$$

which we note is precisely the e_1 term of V^{n-1} in Lemma 3.2.4 on page 16, which is positive.

If $j > m$, then c_{jm} is given by

$$
(-1)^{j+m}
\left|\begin{array}{c|c|c}
\begin{matrix} + & - & \cdots & - \\ - & + & \cdots & - \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \cdots & + \end{matrix} & - & - \\
\hline
- & \overbrace{\begin{matrix} - & - & - & \cdots & - \\ + & - & - & \ddots & \vdots \\ - & + & - & \ddots & - \\ \vdots & \vdots & \ddots & \ddots & - \\ - & \cdots & - & + & - \end{matrix}}^{(j-m)\times(j-m)\text{ block}} & - \\
\hline
- & - & \begin{matrix} + & - & \cdots & - \\ - & + & \cdots & - \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \cdots & + \end{matrix}
\end{array}\right| . \tag{3.2.3}
$$

But note that transposing matrices preserves determinant, and the transpose of the matrix in (3.2.3) is the matrix in (3.2.2), so $c_{jm} = c_{mj}$, which we already know is positive. Therefore, K^{-1} has all positive entries. Since

$$
\begin{pmatrix} -a_{1l} \\ \vdots \\ -a_{nl} \end{pmatrix}
$$

has all positive entries, we define

$$
\begin{pmatrix} b_{1l} \\ \vdots \\ b_{nl} \end{pmatrix} := K^{-1} \begin{pmatrix} -a_{1l} \\ \vdots \\ -a_{nl} \end{pmatrix} .
$$

Define positive measures ν_1, ν_2 by

$$
\nu_l(A) = \mu(A \cap A_l) + \sum_{m=1}^{n} b_{ml}\mu(A \cap B_m) .
$$

Then

$$
\int_B Q_j d\nu_l = a_{jl} + \sum_{m=1}^{n} k_{jm} b_{ml} = 0 ,
$$

so each

$$
h_l = \int_B \mathbb{P}(\cdot, w) d\nu_l(w) , \qquad l = 1, 2 ,
$$

is the real part of an analytic function g_l with $\Im g_l(b) = 0$. We can see that $\nu_1 + \nu_2 = \mu$ as

$$K\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{m=1}^n \int_{B_m} Q_1 d\mu \\ \vdots \\ \sum_{m=1}^n \int_{B_m} Q_n d\mu \end{pmatrix} = \begin{pmatrix} \cancel{P_1(h)} - a_{11} - a_{12} \\ \vdots \\ \cancel{P_n(h)} - a_{n1} - a_{n2} \end{pmatrix}.$$

Multiplying both sides by K^{-1} gives $b_{m1} + b_{m2} = 1$. We therefore have $h_1 + h_2 = h$. Thus, $g_l/g_l(b) \in \mathbb{K}$ and

$$f = g_1(b)\left(\frac{g_1}{g_1(b)}\right) + g_2(b)\left(\frac{g_2}{g_2(b)}\right),$$

so f is a convex combination of two other points in $\mathbb{K}$. Hence, f is not an extreme point. □

Lemma 3.2.8. *The set $\widehat{\mathbb{K}}$ of extreme points of $\mathbb{K}$ is a closed set, and the function taking Π to $\mathbb{K}$ by $p \mapsto f_p$ is a homeomorphism onto $\widehat{\mathbb{K}}$.*

Proof. The proof is essentially that of Lemma 2.11 in [**DM05**], and is included here for completeness.

If we wish to show that the mapping is a homeomorphism, it is sufficient to show that if $p(n)$ converges to $p(0)$, then $f_{p(n)}$ converges to $f_{p(0)}$. Since $\mathbb{K}$ is compact, $f_{p(n)}$ must have at least one accumulation point. Consider any accumulation point f, and choose a subsequence which converges to it, which we denote $f_{p(n_k)}$.

If $h_{p(n)}$ is the real part of $f_{p(n)}$, then we have a representing measure $\mu_n = \sum_j \tau_j(p(n))\delta_{p(n)_j}$ for $h_{p(n)}$, so that

$$h_{p(n)}(z) = \int_B \mathbb{P}(\zeta, z) d\mu_n(\zeta).$$

We know that the measures μ_n converge weakly to μ_0, so $h_{p(n)}$ must converge pointwise to $h_{p(0)}$, and so must $h_{p(n_k)}$, meaning $h_{p(0)}$ must be the real part of f. We conclude that $f_{p(n_k)} \to f_{p(0)}$. This tells us that the only accumulation point of $f_{p(n)}$ must be $f_{p(0)}$, so $f_{p(n)} \to f_{p(0)}$. Since Π is compact, its range, $\widehat{\mathbb{K}}$, must also be compact. □

3.2.4. Test Functions. For $p \in \Pi$, define

$$\psi_p = \frac{f_p - 1}{f_p + 1}.$$

The real part, h_p, of f_p is harmonic across $B \backslash \{p_0, \ldots, p_n\}$, therefore f_p is analytic across $B \backslash \{p_0, \ldots, p_n\}$. Also, $f_p = g_j/(z - p_j)$ in a neighbourhood of p_j, for some analytic g_j, non-vanishing at p_j (by [**Fis83**, Ch. 4, Prop. 6.4]). We can see from this that ψ_p is continuous onto B and $|\psi_p| = 1$ on B.

By the reflection principle, ψ_p is inner and extends analytically across B, and $\psi_p^{-1}\{1\} = \{p_0, \ldots, p_n\}$, so the preimage of each point $z \in \mathbb{D}$ is exactly $n+1$ points, up to multiplicity, and so ψ_p has $n+1$ zeroes.

Similarly, if ψ is analytic in a neighbourhood of R, with modulus one on B and $n+1$ zeroes in R, then $\psi^{-1}\{1\}$ has $n+1$ points. Also, the real part of

$$f = \frac{1+\psi}{1-\psi}$$

is a positive harmonic function which is zero on B except where $\psi(z) = 1$. By Corollary 3.2.2 on page 14, f cannot be identically zero on any B_i, so there must be one point from $\psi^{-1}\{1\}$ on each B_i. If, further, $\psi(b) = 0$, then $\psi = \psi_p$ for some $p \in \Pi$.

Definition 3.2.9. We define $\Theta = \left\{\psi_p : p \in \Pi\right\}$.

Theorem 3.2.10. *If ρ is analytic in R and if $\left|\rho\right| \leq 1$ on R, then there exists a positive measure μ on Π and a measurable function h defined on Π whose values are functions $h(\cdot, p)$ analytic in R so that*

$$1 - \rho(z)\overline{\rho(w)} = \int_\Pi h(z,p)\left[1 - \psi_p(z)\overline{\psi_p(w)}\right]\overline{h(w,p)}d\mu(p)\,.$$

Proof. First suppose $\rho(b) = 0$.

Let

$$f = \frac{1+\rho}{1-\rho}$$

so

$$\rho = \frac{f-1}{f+1}\,.$$

Hence

$$1 - \rho(z)\overline{\rho(w)} = 2\frac{f(z) + \overline{f(w)}}{(f(z)+1)\left(\overline{f(w)}+1\right)}\,. \tag{3.2.4}$$

Since h, the real part of f, is positive and $f(b) = 1$, the function f is in $\mathbb{K}$. Since $\mathbb{K}$ is a compact convex subset of the locally convex topological vector space $\mathcal{H}(R)$, by the Choquet-Bishop-de Leeuw theorem, f is in the closed convex hull of $\widehat{\mathbb{K}} = \{f_p : p \in \Pi\}$, the set of extreme points of $\mathbb{K}$. Therefore, there exists some regular Borel probability measure ν on Π such that

$$f = \int_\Pi f_p d\nu(p)\,. \tag{3.2.5}$$

Using the definition of ψ_p and (3.2.4), we can show that

$$1 - \rho(z)\overline{\rho(w)} = \int_\Pi \frac{1 - \psi_p(z)\overline{\psi_p(w)}}{(f(z)+1)\left(1-\psi_p(z)\right)\left(1-\overline{\psi_p(w)}\right)\left(\overline{f(w)}+1\right)}d\nu(p)\,.$$

Finally, if $\rho(b) = a$, then we have a representation like the one above, as

$$1 - \left(\frac{\rho(z) - a}{1 - \bar{a}\rho(z)}\right)\overline{\left(\frac{\rho(w) - a}{1 - \bar{a}\rho(w)}\right)} = \frac{(1 - a\bar{a})\left(1 - \rho(z)\overline{\rho(w)}\right)}{(1 - \bar{a}\rho(z))\left(1 - a\overline{\rho(w)}\right)}. \tag{3.2.6}$$

□

We conclude that the set Θ is a collection of test functions for $H^\infty(R)$, as defined in Chapter 2.

Note 3.2.11. We have used $n + 1$ parameters to describe the inner functions in Θ, however, we only need n, as we can identify them with the inner functions with $n + 1$ zeroes, by the argument in the introduction to Section 3.2.4 on page 24. If we then fix some $\widetilde{p_0} \in B_0$, it is then clear that for all $p \in \Pi$, $\overline{\psi_p(\widetilde{p_0})}\psi_p$ is an inner function with $n + 1$ zeroes, with one of them at b, and $\overline{\psi_p(\widetilde{p_0})}\psi_p(\widetilde{p_0}) = 1$, so $\overline{\psi_p(\widetilde{p_0})}\psi_p = \psi_q$, where $q = (\widetilde{p_0}, q_1, \dots, q_n)$, for some $q_1 \in B_1, \dots, q_n \in B_n$. We define

$$\widetilde{\Theta} := \left\{\psi_q : q = (\widetilde{p_0}, q_1, \dots, q_n), q_1 \in B_1, \dots, q_n \in B_n\right\},$$

which is also a set of test functions for $H^\infty(R)$.

3.3. Matrix Inner Functions

Recall that $\mathbb{X} = \{x \in R : \varpi(x) = x\}$, that $\mathbb{X}$ has $n + 1$ components $\mathbb{X}_0, \dots,$ $\mathbb{X}_n$, and that these are paths running between the components of B. $\mathbb{X}_i$ runs between $p_i^- \in B_i$ and $p_{i+1}^+ \in B_{i+1}$.

3.3.1. Preliminaries.

Theorem 3.3.1. *If R is symmetric, then there is some $b \in \mathbb{X}$, and some*[1] $\psi_{\mathbf{p}} \in \widetilde{\Theta}$ *with $n + 1$ distinct zeroes $b, z_1, \dots, z_n$, where $z_1, \dots, z_n \notin \mathbb{X}$, and $z_i \neq \varpi(z_j)$ for all i, j.*

Proof. For now, choose a $b_0 \in R$, and use this as our b. We will find a better choice for b later in the proof. Take p_0^- as $\widetilde{p_0}$, and use this to define $\widetilde{\Theta}$ as in Note 3.2.11. We will give this $\widetilde{\Theta}$ an unusual name, $\widetilde{\Theta}_0$, and call the functions in it φ_p, rather than ψ_p. This is to distinguish it from the $\widetilde{\Theta}$ and $\psi_{\mathbf{p}}$ in the statement of the theorem, which we will construct later.

Choose some $p_1 \in B_1 \backslash \mathbb{X}, \dots, p_n \in B_n \backslash \mathbb{X}$. Consider the path v along $\mathbb{X}$ from B_1 to B_0. Its image under φ_p is a path leading to 1. We can see that $\varphi_p^{-1}\{1\}$ has $n + 1$ points. As X is Hausdorff and locally connected, there are

[1] We use $\mathbf{p}$, rather than p here, to indicate that this is a specific $\mathbf{p} \in \Pi$, found in this theorem. When we are dealing with arbitrary elements of Π, we will use p.

disjoint, connected open sets $U_0, U_1, \ldots, U_n$ around each of these points, and since φ_p is an open mapping on each of these open sets,

$$\mathcal{N} := \bigcap_{i=0}^{n} \varphi_p(U_i)$$

is a (relatively) open neighbourhood of 1, whose preimage is $n+1$ disjoint open sets, $U_0', \ldots, U_n'$. Also, we can choose $U_1, \ldots, U_n$ such that none of them intersects $\mathbb{X}$, and none of them intersects any $\varpi(U_i)$ (since $p_1, \ldots, p_n \notin \mathbb{X}$, and $\mathbb{X}$ closed). Now, we can lift $\varphi_p(v) \cap \mathcal{N}$ to each of these U_i', we choose a point $y \in \varphi_p(v) \cap \mathcal{N}$, and note that $\varphi_p^{-1}\{y\}$ has exactly $n+1$ distinct points, none of which maps to another under ϖ, and exactly one of which is on $\mathbb{X}$. The point on $\mathbb{X}$, we use as our b for the rest of the proof. We take a Möbius transform m which preserves the unit circle, and maps y to 0, and notice that $m \circ \varphi_p$ is an inner function which has $n+1$ zeroes, exactly one of which, b, is on $\mathbb{X}$. If we define $\widetilde{\Theta}$ using our new b, and $\widetilde{p_0} = p_0^-$, then $\overline{m \circ \varphi_p(p_0^-)} m \circ \varphi_p \in \widetilde{\Theta}$, and has the required zeroes, and so is our $\psi_{\mathbf{p}}$. □

Remark 3.3.2. Note that in the above argument, we can choose our b as close to p_0^- as we like, so in particular, we can choose b such that $h_0(b) > 1/2$. By an argument similar to that in [**DM05**, Prop. 2.13], we can see that no $\psi_p \in \widetilde{\Theta}$ has all its zeroes at b.

Theorem 3.3.3. *If R is symmetric, then $Q_j(p_i) = \eta(p_i)\, Q_j(\varpi(p_i))$, for some $\eta : B \to \mathbb{C}$ which does not depend on j.*

Proof. We write Q_j as

$$Q_j(p) = \frac{\partial h_j}{\partial n_p}(p)$$

where $\partial/\partial n_p$ is the normal derivative at p. We also define $\partial/\partial t_p$ as the tangent derivative at p.

Now, note that if h is harmonic and ϖ is anticonformal, then $h \circ \varpi$ is also harmonic, and since h_j and $h_j \circ \varpi$ have the same values on B, they must be equal, so

$$\frac{\partial h_j(p_i)}{\partial n_{p_i}} = \frac{\partial h_j(\varpi(p_i))}{\partial n_{p_i}},$$

and so

$$\begin{aligned}Q_j(p_i) =& \frac{\partial h_j(p_i)}{\partial n_{p_i}}\\ =& \frac{\partial h_j(\varpi(p_i))}{\partial n_{\varpi(p_i)}} \cdot \frac{\partial n_{\varpi(p_i)}}{\partial n_{p_i}} + \cancel{\frac{\partial h_j(\varpi(p_i))}{\partial t_{\varpi(p_i)}}} \cdot \frac{\partial t_{\varpi(p_i)}}{\partial n_{p_i}}\\ =& Q_j(\varpi(p_i)) \cdot \underbrace{\frac{\partial n_{\varpi(p_i)}}{\partial n_{p_i}}}_{\eta(p_i)} .\end{aligned}$$

□

Lemma 3.3.4. *If η is defined as above, and $b \in \mathbb{X}$ then*

$$\mathbb{P}(b, p_j) = \eta(p_j)\,\mathbb{P}(b, \varpi(p_j)) .$$

Proof. We can write

$$\mathbb{P}(b, p_j) = \frac{d\omega_b(p_j)}{ds(p_j)} \quad \text{and} \quad \mathbb{P}(b, \varpi(p_j)) = \frac{d\omega_b(\varpi(p_j))}{ds(\varpi(p_j))},$$

and note that if h is harmonic, then $h \circ \varpi$ is harmonic, and $h \circ \varpi(b) = h(b)$. So, for any measurable set $E \subseteq B$,

$$\omega_b(E) = \omega_b(\varpi(E)) ,$$

so $d\omega_b(p_j) = d\omega_b(\varpi(p_j))$. Hence,

$$\begin{aligned}\mathbb{P}(b, p_j) = \frac{d\omega_b(p_j)}{ds(p_j)} = \frac{d\omega_b(\varpi(p_j))}{ds(p_j)} &= \frac{ds(\varpi(p_j))}{ds(p_j)} \cdot \frac{d\omega_b(\varpi(p_j))}{ds(\varpi(p_j))}\\ &= \frac{dn_{\varpi(p_j)}}{dn_{p_j}} \cdot \mathbb{P}(b, \varpi(p_j)) = \eta(p_j)\,\mathbb{P}(b, \varpi(p_j)) ,\end{aligned}$$

since

$$\frac{ds(\varpi(p_j))}{ds(p_j)} \underbrace{=}_{\star} -\frac{dt_{\varpi(p_j)}}{dt_{p_j}} \underbrace{=}_{\dagger} \frac{dn_{\varpi(p_j)}}{dn_{p_j}} ,$$

where $\star$ is due to the fact that ϖ is sense reversing, and $\dagger$ is due to the Cauchy-Riemann equation for anti-holomorphic maps. □

Definition 3.3.5. We say a holomorphic 2×2 matrix valued function F on R has a *standard zero set* if

(1) F has distinct zeroes $b, a_1, \ldots, a_{2n}$, where $F(b) = 0$, and $\det(F)$ has zeroes of multiplicity one at each of $a_1, \ldots, a_{2n}$;
(2) if $\gamma_j \neq 0$ are such that $F(a_j)^*\gamma_j = 0$, $j = 1, \ldots, 2n$, then no $n + 1$ of the γ_j lie on the same complex line through the origin;
(3) $Ja_j \neq P_i$ for $j = 1, \ldots, 2n$, $i = 1, \ldots, n$, where $P_1, \ldots, P_n$ are the poles of the Fay kernel $K^b(\cdot, z)$.

We have not defined K^b yet, and will not do so until Section 3.4. For now, all we need to know about K^b is that all its poles are on $J(\mathbb{X})$.

3.3.2. The construction. We take $\psi_{\mathbf{p}}$ as in Theorem 3.3.1 on page 26. Note that $\overline{\psi_{\mathbf{p}} \circ \varpi}$ is an inner function with zeroes at $b, \varpi(z_1), \ldots, \varpi(z_n)$, equal to one at $p_0^-, \varpi(\mathbf{p}_1), \varpi(\mathbf{p}_2), \ldots, \varpi(\mathbf{p}_n)$, so must equal $\psi_{\varpi(\mathbf{p})}$.

Definition 3.3.6. We say S is a *team of projections* if S is a collection of n pairs of non-zero orthogonal projections on $\mathbb{C}^2$, $\left(P^{j+}, P^{j-}\right)$, such that

$$P^{1+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^{1-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{j+} + P^{j-} = I, \quad j = 1, \ldots n.$$

Let S_0 be the *trivial team*, given by $P^{j\pm} = P^{1\pm}$ for all j.

We define

$$H_{S,p} = \tau_0(p)\mathbb{P}(\cdot, p_0^-)I + \sum_{i=1}^{n} \tau_i(p) \left[\mathbb{P}(\cdot, p_i)\, P^{i+} + \eta(p_i)\mathbb{P}\left(\cdot, \varpi(p_i)\right)\, P^{i-}\right].$$

We note that, by Lemma 3.3.4,

$$\begin{aligned} H_{S,p}(b) &= \tau_0(p)\mathbb{P}(b, p_0^-)I + \sum_{i=1}^{n} \tau_i(p) \left[\mathbb{P}(b, p_i)\, I\right] \\ &= \left[\sum_{i=0}^{n} \tau_i(p)\, \mathbb{P}(b, p_i)\right] I = \cancel{h_p(b)} I = I. \end{aligned}$$

For $x \in \mathbb{C}^2$ a unit vector, $\left\langle H_{S,p}\, x, x\right\rangle$ corresponds to the measure

$$\mu_{x,x} = \tau_0 \delta_{p_0^-} + \sum_{i=1}^{n} \tau_i \cdot \left[\delta_{p_i}\, \left\|P^{i+}x\right\| + \delta_{\varpi(p_i)}\eta(p_i)\, \left\|P^{i-}x\right\|\right],$$

so

$$\begin{aligned} \int_B Q_j d\mu_{x,x} &= \tau_0 Q_j(p_0^-) + \sum_{i=1}^{n} \tau_i \left[Q_j(p_i)\, \left\|P^{i+}x\right\| + \eta(p_i) Q_j\left(\varpi(p_i)\right)\, \left\|P^{i-}x\right\|\right] \\ &= \tau_0 Q_j(p_0^-) + \sum_{i=1}^{n} \tau_i Q_j(p_i) \cancel{\|x\|} \\ &= 0, \end{aligned}$$

by definition of τ.

Hence, $\left\langle H_{S,p}\, x, x\right\rangle$ is the real part of an analytic function, so $H_{S,p}$ is the real part of a holomorphic 2×2 matrix function $G_{S,p}$, normalised by $G_{S,p}(b) = I$.

We now define

$$\Psi_{S,p} = \left(G_{S,p} - I\right) \cdot \left(G_{S,p} + I\right)^{-1}.$$

Lemma 3.3.7. *If $\mathbf{p}$ is as in Theorem 3.3.1 on page 26, for each S:*

(1) $\Psi_{S,\mathbf{p}}$ *is analytic in a neighbourhood of* X *and unitary valued on* B*;*
(2) $\Psi_{S,\mathbf{p}}(b) = 0$*;*
(3) $\Psi_{S,\mathbf{p}}(p_0^-) = I$*;*
(4) $\Psi_{S,\mathbf{p}}(\mathbf{p}_1)e_1 = e_1$ *and* $\Psi_{S,\mathbf{p}}(\varpi(\mathbf{p}_1))\, e_2 = e_2$*;*
(5) $\Psi_{S,\mathbf{p}}(\mathbf{p}_i)\, P^{i+} = P^{i+}$ *and* $\Psi_{S,\mathbf{p}}(\varpi(\mathbf{p}_i))\, P^{i-} = P^{i-}$*;*
(6) $\Psi_{S_0,\mathbf{p}} = \left(\begin{smallmatrix} \psi_{\mathbf{p}} & 0 \\ 0 & \psi_{\varpi(\mathbf{p})} \end{smallmatrix}\right)$.

Proof. Thinking about $\mathbb{P}(z, r)$ as a function of z, in a neighbourhood of $r \in B$, the Poisson kernel $\mathbb{P}(z, r)$ is the real part of some function of the form $g_r(z)(z - r)^{-1}$, where g_r is analytic in the neighbourhood, and non-vanishing at r (by [**Fis83**, Ch. 4, Prop. 6.4]). At any other point $q \in B$, $\mathbb{P}(z, r)$ extends to a harmonic function on a neighbourhood of q, so must be the real part of some analytic function, with real part 0 at q.

We can see that if $r \in B$ is not p_0^-, $\mathbf{p}_1, \ldots, \mathbf{p}_n, \varpi(\mathbf{p}_1), \ldots, \varpi(\mathbf{p}_n)$, then $G_{S,\mathbf{p}}$ is analytic in a neighbourhood of r. Further, $G_{S,\mathbf{p}} + I$ is invertible near r as $G_{S,\mathbf{p}}(z) = H_{S,\mathbf{p}}(z) + iA(z)$ for some self-adjoint matrix valued function $A(z)$, and $H_{S,\mathbf{p}}(r) = 0$. Thus, $G_{S,\mathbf{p}} + I$ is invertible at and, by continuity, near r. We have

$$I - \Psi_{S,\mathbf{p}}\Psi_{S,\mathbf{p}}^* = 2(G_{S,\mathbf{p}} + I)^{-1} \underbrace{(G_{S,\mathbf{p}} + G_{S,\mathbf{p}}^*)}_{iA+(iA)^*=0}(G_{S,\mathbf{p}} + I)^{*-1},$$

which is zero at r, so $\Psi_{S,\mathbf{p}}$ must be unitary at r.

From the definition of $G_{S,\mathbf{p}}$, in a neighbourhood of p_0^-, there are analytic functions g_1, g_2, k_1, k_2 so that the real parts of k_j are 0 at p_0^-, each g_j is non-vanishing at p_0^-, and

$$G_{S,\mathbf{p}}(z) = \begin{pmatrix} \frac{g_1(z)}{z-p_0^-} & k_1(z) \\ k_2(z) & \frac{g_2(z)}{z-p_0^-} \end{pmatrix},$$

so

$$\begin{aligned}(G_{S,\mathbf{p}}(z) + I)^{-1} &= \frac{1}{\frac{g_1+z-p_0^-}{z-p_0^-}\frac{g_2+z-p_0^-}{z-p_0^-} - k_1(z)k_2(z)} \begin{pmatrix} \frac{g_2(z)-z-p_0^-}{z-p_0^-} & -k_1(z) \\ -k_2(z) & \frac{g_1(z)-z-p_0^-}{z-p_0^-} \end{pmatrix} \\ &= \frac{\begin{pmatrix} \left(g_2(z) - z - p_0^-\right)(z - p_0^-) & -k_1(z)(z - p_0^-)^2 \\ -k_2(z)(z - p_0^-)^2 & \left(g_1(z) - z - p_0^-\right)(z - p_0^-) \end{pmatrix}}{\left(g_1(z) - z - p_0^-\right)\left(g_2(z) - z - p_0^-\right) - k_1(z)k_2(z)\,(z - p_0^-)^2}.\end{aligned}$$

Note that the denominator is non-zero at and near p_0^-, so $G_{S,\mathbf{p}} + I$ is invertible. We can use this to calculate $\Psi_{S,\mathbf{p}}$ directly[2], and show that $\Psi_{S,\mathbf{p}}$ is analytic in a neighbourhood of p_0^-, and $\Psi_{S,\mathbf{p}}(p_0^-) = I$, so we have (3).

[2]The calculation is omitted, but can be readily verified by hand, or with a computer algebra system

Now we look at $\mathbf{p}_1$. Near $\mathbf{p}_1$ we have analytic functions g, k_1, k_2, k_3, on a neighbourhood of $\mathbf{p}_1$, where k_1, k_2, k_3 have zero real part at $\mathbf{p}_1$, g is non-zero at $\mathbf{p}_1$, and

$$G_{S,\mathbf{p}}(z) = \begin{pmatrix} \frac{g(z)}{z-\mathbf{p}_1} & k_1 \\ k_2 & k_3 \end{pmatrix}.$$

Since $k_3 + 1$ has real part 1 at $\mathbf{p}_1$, $g(z)\,(z - \mathbf{p}_1)^{-1}$ has a pole, and k_1, k_2 are analytic at $\mathbf{p}_1$, we see that $G_{S,\mathbf{p}}+I$ is invertible near $\mathbf{p}_1$. By direct computation, we see that $\Psi_{S,\mathbf{p}}$ is analytic in a neighbourhood of $\mathbf{p}_1$ and

$$\Psi_{S,\mathbf{p}}(\mathbf{p}_1) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k_3(\mathbf{p}_1)-1}{k_3(\mathbf{p}_1)+1} \end{pmatrix}.$$

A similar argument holds for $\varpi(\mathbf{p}_1)$, so we have (4), and by working in the orthonormal basis induced by P^{j+} and P^{j-}, (5) follows. Also, we have now shown $\Psi_{S,\mathbf{p}}$ is analytic at every point, so (1) follows.

(6) and (2) follow easily from the definitions. □

LEMMA 3.3.8. *We define* $\|S_1 - S_2\|_\infty = \max_{j\pm} \left\| P_1^{j\pm} - P_2^{j\pm} \right\|$, *giving a metric on the space* $\mathcal{T}$ *of all teams of projections. There exists some non-trivial sequence* $S_m \to S_0$ *such that for all* m, $\Psi_{S_m,\mathbf{p}}$ *has a standard zero set.*

PROOF. Since the zeroes of $\psi_{\mathbf{p}}$ and $\psi_{\varpi(\mathbf{p})}$ are all distinct except for b, it is clear that

$$\Psi_{S_0,\mathbf{p}} = \begin{pmatrix} \psi_{\mathbf{p}} & 0 \\ 0 & \psi_{\varpi(\mathbf{p})} \end{pmatrix}$$

has a standard zero set.

We note that whatever value we take for ϵ, there is an $S \neq S_0$ within ϵ of S_0, so there is some non-trivial sequence S_m converging to S_0.

The sequence $\Psi_{S_m,\mathbf{p}}$ is uniformly bounded, so has a sub-sequence Ψ_m which converges uniformly on compact subsets of R to some Ψ. This means

$$G_m = (I + \Psi_m)(I - \Psi_m)^{-1}$$

converges uniformly on compact subsets of R to

$$G = (I + \Psi)(I - \Psi)^{-1}.$$

H_m, the real part of G_m is harmonic, and

$$H_m - H_0 = \sum_{i=2}^{n} \tau_i(\mathbf{p})\mathbb{P}(\cdot,\, \mathbf{p}_i)\left[P_m^{i+} - P^{1+}\right] + \tau_i\left(\varpi(\mathbf{p})\right)\mathbb{P}(\cdot,\, \varpi(\mathbf{p}_i))\left[P_m^{i-} - P^{1-}\right].$$

Since $P_m^{i\pm} \to P^{1\pm}$, we see that $H_m \to H_0$, and since $G(b) = I = G_0(b)$, $G_m \to G_0$, so $\Psi = \Psi_0$, and $\Psi_m \to \Psi_0$ uniformly on compact sets.

Let $d_m(z) = \det(\Psi_m(z))$. This is analytic, and unimodular on B. Draw small, disjoint circles in R around the zeroes of d_0 (which correspond to the zeroes of Ψ_0). By Hurwitz's theorem, there exists some M such that for all $m \geq M$, d_m and d_0 have the same number of zeroes in each of these circles, so the zeroes of d_m must be distinct, apart from the repeated zero at b. In particular, the zeroes $(b, a_1^m, \ldots, a_{2n}^m)$ of Ψ_m converge to the zeroes $(b, a_1^0, \ldots, a_{2n}^0)$ of Ψ_0.

Finally, if $\|\gamma_1^m\| = 1$, $\Psi_m(a_1^m)^*\gamma_1^m = 0$ and a_1^m is close to a_1^0, then

$$\Psi_0(a_1^0)^*\gamma_1^m = \left(\Psi_0(a_1^0) - \Psi_0(a_1^m)\right)^* \gamma_1^m + \left(\Psi_0(a_1^m) - \Psi_m(a_1^m)\right)^* \gamma_1^m .$$

However, the right hand side tends to zero as m tends to infinity, so $\Psi_0(a_1^0)^*\gamma_1^m$ tends to zero. Since γ_1^m is a bounded sequence in a finite-dimensional complex space, it has a convergent sub-sequence, which we shall also call γ_1^m. This γ_1^m must converge to something in the kernel of $\Psi_0(a_1^0)^*$, that is, a multiple of e_1. We apply this argument to $a_2, \ldots, a_{2n}$, and find a sub-sequence such that n of the γ_i^ms tend to multiples of e_1 and n of them tend to multiples of e_2, so for m big enough, no $n+1$ of them are collinear. □

3.4. Theta Functions

3.4.1. The Jacobian Variety. We know that for each $i = 1, \ldots, n$, h_i is locally the real part of an analytic function g_i. The differential dg_i can be extended from R to Y (as in Theorem 3.1.1, Y is the Schottky double of R), and

$$\alpha_i := \frac{1}{2} dg_i\,, \qquad i = 1, \ldots, n$$

is then a basis for the space of holomorphic 1-forms on Y. We see that if we define a homology basis for Y by $A_j = \mathbb{X}_j - J(\mathbb{X}_j)$ and B_j as before, then $\int_{A_j} \alpha_i = \delta_{ij}$ and

$$\Omega := \left(\int_{B_j} \alpha_i \right)_{ij}$$

has positive definite imaginary part (see, for example, [**FK92**, III.2.8]).

We define a lattice

$$L := \mathbb{Z}^n + \Omega\mathbb{Z}^n \subseteq \mathbb{C}^n\,,$$

define the *Jacobian variety* by

$$\mathcal{J}(Y) := \mathbb{C}^n / L\,,$$

and define the *Abel-Jacobi maps* $\chi : Y \to \mathbb{C}^n$ and $\chi_0 : Y \to \mathcal{J}(Y)$ by

$$\chi(y) := \begin{pmatrix} \int_{p_0^-}^{y} \alpha_1 \\ \vdots \\ \int_{p_0^-}^{y} \alpha_n \end{pmatrix}, \qquad \chi_0(y) = [\chi(y)]\,.$$

Note that the integral depends on the path of integration. However, any two paths differ only by a closed path, and $A_1, \ldots, A_n, B_1, \ldots, B_n$ is a homology basis for Y, so any closed path is homologous to a sum of paths in this basis. Also,

$$\int_{A_j} \alpha_i\,, \int_{B_j} \alpha_i \in L, \text{ so } \left[\int_{A_j} \alpha_i\right] = \left[\int_{B_j} \alpha_i\right] = 0\,,$$

so the particular choice of path we integrate over does not affect $\chi_0(y)$.

Proposition 3.4.1. *The Abel-Jacobi map has the following properties:*

(1) χ_0 *is a one-one conformal map of* Y *onto its image in* $\mathcal{J}(Y)$*; and*
(2) $\chi_0(Jy) = -\chi_0(y)^*$*, where* * *denotes the coordinate-wise conjugate.*

Proof. (1) is proved in [**FK92**, III.6.1], (2) holds because $p_0^- \in \mathbb{X}$ and

$$g_j(Jy) - g_j(p_0^-) = -\overline{\left(g_j(y) - g_j(p_0^-)\right)}\,.$$

□

3.4.2. Theta Functions.

Definition 3.4.2. Roughly following [**Mum83**], we define the *theta function* $\vartheta : \mathbb{C}^n \to \mathbb{C}$ by

$$\vartheta(z) = \sum_{m \in \mathbb{Z}^n} \exp\left(\pi i \langle \Omega m, m\rangle + 2\pi i \langle z, m\rangle\right)\,,$$

where $\langle \cdot, \cdot \rangle$ is the usual $\mathbb{C}^n$ inner product. This function is quasi-periodic, as

$$\begin{aligned} \vartheta(z + m) &= \vartheta(z) \\ \vartheta(z + \Omega m) &= \exp\left(-\pi i \langle \Omega m, m\rangle - 2\pi i \langle z, m\rangle\right) \vartheta(z) \end{aligned}$$

for all $m \in \mathbb{Z}^n$, as shown in [**Mum83**, p. 120]. Given $e \in \mathbb{C}^n$, we rewrite this as $e = u + \Omega v$ for some $u, v \in \mathbb{R}^n$, and we define the *theta function with characteristic* e, $\vartheta[e] : \mathbb{C}^n \to \mathbb{C}$ by

$$\vartheta[e](z) = \vartheta\begin{bmatrix} u \\ v \end{bmatrix}(z) = \exp\left(\pi i \langle \Omega v, v\rangle + 2\pi i \langle z + u, v\rangle\right) \vartheta(z + e)\,.$$

Note that this follows [**Mum83**, p. 123]. Subtly different definitions are used in [**Fay73**], [**DM05**] and [**FK92**], although these differences are not particularly important.

Theorem 3.4.3. *There exists a constant vector* Δ *(called the* vector of Riemann constants*), depending on the choice of base-point, such that for each* $e \in \mathbb{C}^n$*, either* $\vartheta[e] \circ \chi$ *is identically zero, or* $\vartheta[e] \circ \chi$ *has exactly* n *zeroes,* $\zeta_1, \ldots, \zeta_n$ *and*

$$\sum_{i=1}^{n} \chi(\zeta_i) = \Delta - e\,.$$

Proof. See [**Mum83**, Ch. 2, Cor. 3.6] or [**FK92**, VI.2.4]. □

For the following, it will be convenient to define

$$\mathcal{E}_e(x, y) = \vartheta(\chi(y) - \chi(x) + e)\,.$$

Theorem 3.4.4. *If $e \in \mathbb{C}^n$, $\vartheta(e) = 0$ and $\mathcal{E}_e$ is not identically zero, then there exist $\zeta_1, \ldots, \zeta_{n-1}$ such that for each $x \in Y$, $x \neq \zeta_i$, the zeroes of $\vartheta[e - \chi(x)] \circ \chi$, which coincide with the zeroes of $\mathcal{E}_e(x, \cdot)$, are precisely $x, \zeta_1, \ldots, \zeta_{n-1}$.*

Proof. See [**Mum83**, Ch. 2, Lemma 3.4]. □

Theorem 3.4.5. *There exists an $e_* = u_* + \Omega v_* \in \mathbb{C}^n$ such that $2e_* = 0$ mod L, $\langle u_*, v_*\rangle$ is an odd integer, and $\mathcal{E}_{e_*} \not\equiv 0$.*

For the proof see [**Mum84**, Ch. IIIb, Sec. 1, Lemma 1], although the remarks at the end of [**FK92**, VI.1.5] provide some relevant discussion. An e_* of this type is called a *non-singular odd half-period*, and we see that $\vartheta[e_*]$ is an odd function, so $\vartheta(e_*) = 0$.

Let $\vartheta_* := \vartheta[e_*]$, so

$$\vartheta_*(t) = \exp\left(\pi i \langle \Omega v_*, v_*\rangle + 2\pi i \langle z + u_*, v_*\rangle\right) \vartheta(z + e_*)\,.$$

Clearly, we can apply Theorems 3.4.4 and 3.4.5, and get that the roots of

$$\vartheta_*\left(\chi(\cdot) - \chi(z)\right)$$

are $\{z, \zeta_1, \ldots, \zeta_{n-1}\}$ for some $\zeta_1, \ldots, \zeta_{n-1}$. If neither of $z, w \in Y$ coincide with with any of these ζ_is, then

$$\frac{\vartheta_*\left(\chi(\cdot) - \chi(z)\right)}{\vartheta_*\left(\chi(\cdot) - \chi(w)\right)} = e^{2\pi i \langle w - z, v_*\rangle} \frac{\vartheta\left(\chi(\cdot) - \chi(z) + e_*\right)}{\vartheta\left(\chi(\cdot) - \chi(w) + e_*\right)} \tag{3.4.1}$$

is a multiple valued function with exactly one zero and one pole, at z and w respectively.

3.4.3. The Fay Kernel. A tool that will prove invaluable in later sections is the Fay kernel K^a, which is a reproducing kernel on $H^2(R, \omega_a)$, the Hardy space of analytic functions on R with boundary values in $L^2(\omega_a)$. For a more comprehensive discussion of the ideas in this section, see [**Fay73**].

Lemma 3.4.6. *The critical points of the Green's function $g(\cdot, b)$ are on $\mathbb{X}$, one in each $\mathbb{X}_i$, $i = 1, \ldots, n$.*

Proof. We write $g(z) = g(z, b)$. We know that g has n critical points, by [**Neh52**, p. 133-135]. On $\mathbb{X}$, define $\partial/\partial x$ as the derivative tangent to $\mathbb{X}$, and $\partial/\partial y$ as the derivative normal to $\mathbb{X}$. We know that $g \circ \varpi = g$, and

$$\frac{\partial g}{\partial y} = \frac{\partial g \circ \varpi}{\partial y} = \frac{\partial g}{\partial y} \cdot \frac{\partial \varpi_y}{\partial y} + \frac{\partial g}{\partial x} \cdot \cancel{\frac{\partial \varpi_x}{\partial y}}.$$

However, $\partial\varpi_y/\partial y < 0$ on $\mathbb{X}$, so the two sides of this equation have different signs, and so $\partial g/\partial y = 0$ on $\mathbb{X}$. Also, $g = 0$ on B, so g must be zero at p_i^- and p_{i+1}^+ – the start and end points of $\mathbb{X}_i$. Since $\partial g/\partial x$ is continuous on $\mathbb{X}_i$ (provided $i \neq 0$), $\partial g/\partial x$ must be zero somewhere on $\mathbb{X}_i$, by Rolle's theorem. Since this gives us n distinct zeroes, this must be all of them. ◻

We have just proved that $g(\cdot, b)$ has n distinct zeroes. If these zeroes are $z_1 \in \mathbb{X}_1, \ldots, z_n \in \mathbb{X}_n$ we define $P_i = Jz_i$.

Theorem 3.4.7. *There is a reproducing kernel K^b for the Hardy space $H^2(R, \omega_b)$; that is, if $f \in H^2(R, \omega_b)$, then*

$$f(a) = \left\langle f(\cdot), K^b(\cdot, a)\right\rangle := \int_{\partial R} f(z)\overline{K^b(z, a)}d\omega_b(z)\,.$$

If $a = b$, then $K^b(\cdot, a) \equiv 1$. If not, $K^b(\cdot, a)$ has precisely the poles

$$P_1(b), \ldots, P_n(b), Ja$$

(where $JP_1(b), \ldots, JP_n(b)$ are the critical points of $g(\cdot, b)$), and $n + 1$ zeroes in Y, one of which is Jb.

Sketch Proof. The precise definition of K^a we will use, is that found in [**Fay73**, Prop. 6.15]. The author defines a point $e \in \mathcal{J}(Y)$ (definition not repeated here), and gives

$$K^a(x, y) = \tag{3.4.2}$$
$$\frac{\vartheta\left(\chi(x) + \chi(y)^* + e\right)\vartheta\left(\chi(b) + \chi(b)^* + e\right)\vartheta_*\left(\chi(b) + \chi(y)^*\right)\vartheta_*\left(\chi(x) + \chi(b)^*\right)}{\vartheta\left(\chi(b) + \chi(y)^* + e\right)\vartheta\left(\chi(x) + \chi(b)^* + e\right)\vartheta_*\left(\chi(x) + \chi(y)^*\right)\vartheta_*\left(\chi(b) + \chi(b)^*\right)}\,.$$

This K^b will turn out to be the the reproducing kernel[3] for $H^2(R, \omega_b)$.

To see why this is, we must first understand a little more about ω_b. In many ways, it is easier to understand the generalisation of the result, given in [**BC96**]. The authors in that paper were interested in *representing measures* – that is, measures μ such that

$$\int_{\partial R} f(z)d\mu(z) = f(b)$$

for all rational functions f. We know that the harmonic measure, ω_b is such a measure, as all rational functions are harmonic.

It is shown in [**Cla91**] that every representing measure corresponds to a meromorphic differential on Y, so for any representing measure μ, there is a meromorphic differential dw such that $d\mu = dw|_{\partial R}$. Moreover,

[3]Fay gives (3.4.2) in a slightly different form, although we can use [**Fay73**, Prop. 6.1] and some basic results on theta functions to show that the two forms are equivalent. Also note that the notation Fay uses differs significantly from the notation used here.

the meromorphic differential corresponding to ω_b is found by taking the Green's function $g(\cdot, b)$, locally pairing it with its harmonic conjugate to give a locally defined meromorphic function, analytically continuing it onto Y, and taking the derivative of this function to give dw_b. A consequence of this construction is that dw_b is zero precisely at the critical points of $g(\cdot, b)$ (and their mirror images under J) and has poles at b and Jb.

The fact that the zeroes and poles of $K^b(\cdot, z)$ are as stated, is an essential part of the construction of K^b, found in either **[BC96]** or **[Fay73]**. The argument used to construct K^b is too lengthy to include here.

However, once we know these facts about the zeroes and poles of K^b, we can show that it is a reproducing kernel. We can write

$$\begin{aligned}\left\langle f, K^b(\cdot, a)\right\rangle &= \int_{\partial R} f(z)\overline{K^b(z, a)}d\omega_b(z) \\ &= \int_{\partial R} f(z)\overline{K^b(z, a)}dw_b(z) \\ &= \int_{\partial R} f(z)K^b(Jz, a)dw_b(z)\,.\end{aligned}$$

If we consider the differential $K^b(Jz, a)dw_b(z)$, we see that its only pole on R is at a ($K^b(Jz, a)$ has poles at the $JP_1(b), \ldots, JP_n(b)$, but these cancel with the zeroes of $dw_b(z)$. Equally, whilst $dw_b(z)$ has a pole at b, this cancels with a zero of $K^b(Jz, a)$). This means that we can apply the residue theorem, to see that

$$\int_{\partial R} f(z)K^b(Jz, a)dw_b(z) = f(a)\,.$$

□

We will write $P_i(b) = P_i$, for brevity.

Theorem 3.4.8. *Let $a_1^0, \ldots, a_{2n}^0$ be points in R such that*

$$P_1, \ldots, P_n, Jb, Ja_1^0, \ldots, Ja_{2n}^0$$

are all distinct. Let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{C}^2$ and let

$$\gamma_1^0 = \cdots = \gamma_n^0 = e_1\,, \quad \gamma_{n+1}^0 = \cdots = \gamma_{2n}^0 = e_2\,.$$

There exists an $\epsilon > 0$ so that if $\left|a_j^0 - a_j\right|, \left\|\gamma_j^0 - \gamma_j\right\| < \epsilon$, and

$$h(z) = \sum_{j=1}^{2n} c_j K^b(z, a_j)\gamma_j + v \tag{3.4.3}$$

is a $\mathbb{C}^2$-valued meromorphic function which does not have poles at $P_1, \ldots, P_n$, then h is constant; that is, each $c_j = 0$.

Further, if $h \neq 0$ has a representation as in (3.4.3)*, and there exists $z' \in R\backslash\{b\}$ such that*

$$h(z)K^b(z, z') = \sum c'_j K^b(z, a_j)\gamma_j + v'$$

then h is constant, $z' = a_j$ for some j, $c'_j\gamma_j = h$, and all other terms are zero.

This theorem can be seen as a result about meromorphic functions on Y, so we view z as a local co-ordinate on Y. If we are only interested in values of z near one of $P_1, \ldots, P_n$, we can assume $z, P_1, \ldots, P_n, Ja_1, \ldots, Ja_{2n}$ are in a single chart $U \subseteq J(R)$ (U is open and simply connected).

A useful tool in the proof of this theorem is the residue of K^b. We know that so long as $a \notin \{b, P_1, \ldots, P_n\}$, $K^b(\cdot, a)$ has only simple poles, so we know that in a small enough neighbourhood of P_j,

$$(z - P_j)K^b(z, a)$$

is a holomorphic function in z. Let $R_j(a)$ denote the value of this function at P_j.

We will need the following lemma.

Lemma 3.4.9. *The residue $R_j(a)$ varies continuously with a.*

Proof. Consider the theta function representation of $K^b(z, a)$. The function

$$f(z) = \vartheta\,(\chi(z) + \chi(b)^* + e)$$

is analytic and single valued on U, and vanishes with order one at P_j, so can be written as

$$f(z) = (z - P_j)f_j(z)$$

for some f_j analytic on U, and non-vanishing at P_j. Given a set $W \subseteq U$, let $W^* = \{\overline{z} : z \in W\}$. Choose neighbourhoods V_j, W of U so that $F : V_j \times W^* \to \mathbb{C}$ given by

$$\begin{aligned} F(z, a) &= f(z)K^b(z, a) \\ &= \frac{\vartheta\,(\chi(z) + \chi(a)^* + e)\,\vartheta\,(\chi(b) + \chi(b)^* + e)\,\vartheta_*\,(\chi(b) + \chi(a)^*)\,\vartheta_*\,(\chi(z) + \chi(b)^*)}{\vartheta\,(\chi(b) + \chi(a)^* + e)\,\vartheta_*\,(\chi(z) + \chi(a)^*)\,\vartheta_*\,(\chi(b) + \chi(b)^*)} \end{aligned}$$

is analytic in $(z, \overline{a})$. Rewriting gives

$$(z - P_j)K^b(z, a) = \frac{F(z, a)}{f_j(z)}\,.$$

The lemma follows from the fact that the right hand side is analytic in $(z, \overline{a})$. □

We can now prove Theorem 3.4.8.

PROOF OF THEOREM 3.4.8. We can assume ϵ is small enough that

$$P_1, \dots, P_n, Ja_1, \dots, Ja_{2n}$$

are distinct. We define

$$\mathfrak{R}_1 = \begin{pmatrix} R_1(a_1) & \cdots & R_1(a_n) \\ \vdots & \ddots & \vdots \\ R_n(a_1) & \cdots & R_n(a_n) \end{pmatrix}$$

and

$$\mathfrak{R}_2 = \begin{pmatrix} R_1(a_{n+1}) & \cdots & R_1(a_{2n}) \\ \vdots & \ddots & \vdots \\ R_n(a_{n+1}) & \cdots & R_n(a_{2n}) \end{pmatrix},$$

where $R_j(a)$ is the residue of $K^b(\cdot, a)$ at P_j, as before.

To see that $\mathfrak{R}_1$ is invertible, let

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

and

$$f_c = \sum_{j=1}^{n} c_j K^b(\cdot, a_j).$$

Note that $\mathfrak{R}_1 c = 0$ if and only if f_c does not have poles at any P_j. Now, if this is the case, then f_c can only have poles at $Ja_1, \dots, Ja_n$, and simple poles at that, but this is only n points, so by Proposition 3.1.4, f_c must be constant. We know that $K^b(\cdot, b) = 1$, so we can say that

$$0 = c_0 K^b(\cdot, b) + c_1 K^b(\cdot, a_1) + \cdots + c_n K^b(\cdot, a_n).$$

However, we know that $K^b(\cdot, b)$, $K^b(\cdot, a_1), \dots, K^b(\cdot, a_n)$ are linearly independent, so $c = 0$. Therefore $\mathfrak{R}_1$ is invertible, and by a similar argument $\mathfrak{R}_2$ is invertible.

Now, consider the function F defined for γ_j near γ_j^0 by

$$F = \begin{pmatrix} R_1(a_1)\gamma_1 & \cdots & \cdots & R_1(a_{2n})\gamma_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ R_2(a_1)\gamma_1 & \cdots & \cdots & R_2(a_{2n})\gamma_{2n} \end{pmatrix}.$$

We define F_0 similarly, using a_j^0 and γ_j^0. We can see that F is an $n \times 2n$ matrix with entries from $\mathbb{C}^2$, so can be regarded as a $2n \times 2n$ matrix. We know that F varies continuously with each γ_j, and by Lemma 3.4.9, varies continuously with each a_j. Also, we see that, by regarding F_0 as a $2n \times 2n$ matrix, the rows of F_0 can be shuffled to give

$$\begin{pmatrix} \Re_1 & 0 \\ 0 & \Re_2 \end{pmatrix}$$

which is invertible, so F_0 is invertible. We can therefore choose $\epsilon > 0$ small enough that if $\left|a_j - a_j^0\right|, \left\|\gamma_j - \gamma_j^0\right\| < \epsilon$ for all j, then F is invertible.

If the a_j and γ_j are chosen such that F is invertible and

$$h(z) = \sum_{j=1}^{n} c_j K^b(z, a_j)\gamma_j + v$$

does not have poles at P_j, then

$$0 = \begin{pmatrix} \sum_{j=1}^n c_j R_1(a_j)\gamma_j \\ \vdots \\ \sum_{j=1}^n c_j R_n(a_j)\gamma_j \end{pmatrix} = F \begin{pmatrix} c_1 \\ \vdots \\ c_{2n} \end{pmatrix} = Fc\,,$$

so $c = 0$, and h is constant.

Now we prove the second part of the theorem. Note that the proof of this part only assumes that the result of the first part holds, not the assumptions on a_j and γ_j used to prove it. Suppose $h \neq 0$ and there exists $z' \in R\backslash\{b\}$ such that

$$h(z)K^b(z, z') = \sum c_j' K^b(z, a_j)\gamma_j + v'\,.$$

We can see that $P_1, \ldots, P_n$ are not poles of h, since by the assumptions on the distinctness of the P_ks and a_js, the right hand side has a pole of order at most one at each P_k, whilst the left hand side has poles of order at least one at each of these points. Therefore, since h has a representation as in the first part of the theorem, h is constant. □

3.5. Representations

This section inherits much of its structure from [**DM05**], and in particular, the results in this section are analogues of results from that paper. In fact, in some cases, the proofs in [**DM05**] do not use the connectivity of X, so can be used to prove their analogues here simply by noting this fact. The proofs are included only for completeness.

3.5.1. Kernels, Realisations and Interpolation. We note, for those who are interested, that many of these results have a similar flavour to some of the Schur-Agler class results from [**DM07**], although we shall not use any of these results directly.

The following two results form a major part of the proof of Theorem 3.0.5 on page 11. Definitions can be found there.

LEMMA 3.5.1. *If $F \in M_2(H^\infty(X))$, then there exists a $\rho > 0$ such that*

$$I - \rho^2 F(z)F(w)^* \in \mathcal{C}.$$

PROOF. Initially, we assume that

$$F = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is, we assume F has only one column (it has two). Choose a $\tau > 0$ large enough that f/τ and g/τ are both in $\mathcal{B}H^\infty(X)$. We now see that

$$\begin{aligned} 2\tau^2 - F(z)F(w)^* &= 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(\tau^2 - f(z)\overline{f(w)}\right)\begin{pmatrix} 1 & 0 \end{pmatrix} \\ &+ 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(\tau^2 - g(z)\overline{g(w)}\right)\begin{pmatrix} 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} f(z) \\ -g(z) \end{pmatrix}(1-0)\begin{pmatrix} \overline{f(w)} & -\overline{g(w)} \end{pmatrix}, \end{aligned}$$

and so $1 - \frac{1}{2\tau^2}F(z)F(w)^* \in \mathcal{C}$.

We now deal with both columns of F. If F has two columns, then

$$F(z)F(w)^* = G(z)G(w)^* + H(z)H(w)^*,$$

where G and H are the columns of F. We know there must be ρ_G and ρ_H such that

$$1 - \rho_G^2 G(z)G(w)^*,\ 1 - \rho_H^2 H(z)H(w)^* \in \mathcal{C}.$$

Since $\mathcal{C}$ is a cone, we know that

$$\left(\frac{1}{\rho_G^2} + \frac{1}{\rho_H^2}\right) - F(z)F(w)^* = \left(\frac{1}{\rho_G^2} - G(z)G(w)^*\right) + \left(\frac{1}{\rho_H^2} - H(z)H(w)^*\right) \in \mathcal{C}.$$

□

THEOREM 3.5.2. *If there is a function $F : R \to M_2(\mathbb{C})$ which is analytic in a neighbourhood of X and unitary valued on B, such that $\rho_F < 1$, then there exists an operator $T \in \mathcal{B}(H)$ for some Hilbert space H, such that the homomorphism $\pi : H^\infty(X) \to \mathcal{B}(H)$ given by the holomorphic functional calculus is contractive, but not completely contractive.*

PROOF. We use cone separation with a GNS construction.

By the hypothesis, we can choose a $\rho < 1$ such that $I - \rho^2 F(z)F(w)^* \notin \mathcal{C}$. We use this ρ for the remainder of the proof.

We define a vector space $\mathcal{P}$, containing all finite sums of the form

$$\sum_j h_j(z)g_j(w)^*$$

where h_j and g_j are $\mathbb{C}^2$-valued analytic functions on a neighbourhood of X. This means that all functions in $\mathcal{P}$ are 2×2 matrix valued. We can see that the cone C sits inside $\mathcal{P}$.

Consider the real vector space $\mathfrak{C}$ spanned by C. Both $I - \rho^2 F(z)F(w)^*$ and C sit within $\mathfrak{C}$, and by Lemma 3.5.1, I is in the algebraic interior of C. We can therefore apply the Basic Separation Theorem from **[Hol75]**, to see that there is a non-zero linear functional $\lambda_{\mathfrak{C}}$ on $\mathfrak{C}$, such that $\lambda_{\mathfrak{C}} \geq 0$ on C, $\lambda_{\mathfrak{C}}(I) > 0$ and $\lambda_{\mathfrak{C}}(1 - \rho^2 F(z)F(w)^*) \leq 0$. Since $\mathfrak{C} \subseteq \mathcal{P}$, we can choose an extension of $\lambda_{\mathfrak{C}}$ to $\mathcal{P}$, which we call λ.

We write $\mathbb{H}_2(X)$ for the space of $\mathbb{C}^2$-valued functions analytic on a neighbourhood of X. We define a pre-inner product on $\mathbb{H}_2(X)$ by

$$[h, g] = \lambda \left(h(z)g(w)^* \right) .$$

Since $h(z)h(w)^* \in C$, $[\cdot, \cdot]$ is positive semi-definite (this is what we mean by a pre-inner product).

Given a function f analytic on a neighbourhood of X, we define a linear map $M_f : \mathbb{H}_2(X) \to \mathbb{H}_2(X)$ by $M_f g = fg$. We let $C_f := \sup_{x \in X} f(x)$, so $\left| f/c \right| < 1$ for all $c > C_f$.

For any $g \in \mathbb{H}_2(X)$,

$$\left(C_f g(z)\right)\left(1 - \frac{f(z)}{C_f}\frac{\overline{f(w)}}{C_f}\right)\left(C_f g(w)\right)^* = g(z)\left(C_f^2 - f(z)\overline{f(w)}\right) g(w)^* \in C .$$

Therefore,

$$\begin{aligned} C_f^2 [g, g] - \left[M_f g, M_f g\right] &= C_f^2 \lambda \left(g(z)g(w)^* \right) - \lambda \left(f(z)g(z)g(w)^* f(w)^* \right) \\ &= \lambda \left(g(z)\left(C_f^2 - f(z)\overline{f(w)}\right) g(w)^* \right) \geq 0 , \end{aligned}$$

since $\lambda \geq 0$ on C. We can see that,

$$C_f^2 [g, g] \geq \left[M_f g, M_f g\right] .$$

Firstly, this tells us that $\left[M_f g, M_f g\right] = 0$ whenever $[g, g] = 0$, so M_f takes $[\cdot, \cdot]$-null vectors to $[\cdot, \cdot]$-null vectors. If we mod $\mathbb{H}_2(X)$ out by $[\cdot, \cdot]$-null vectors to get an inner product space

$$\mathcal{H} := \mathbb{H}_2(X) / \ker[\cdot, \cdot] ,$$

then M_f is a linear map on $\mathcal{H}$. Secondly, this tells us that M_f is a bounded linear map on $\mathcal{H}$ – bounded by C_f. Since M_f is continuous, we can extend it continuously to the Hilbert space completion of $\mathcal{H}$, which we continue to denote by $\mathcal{H}$.

The function $\zeta(z) = z$ is an analytic function on X, so we can define $T = M_\zeta$. This will be our counterexample.

We can see that $f(T) = M_f$, so

$$\left\|f(T)\right\| = \left\|M_f\right\| \leq C_f = \left\|f\right\|_{H^\infty(X)} .$$

Therefore, the homomorphism π is contractive. To see that π is not completely contractive, we show that $\|\pi_2(F)\| > 1$, even though $\|F\| = 1$.

We will write $F(T)$ for $\pi_2(F)$. We write

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

so

$$F(T) = \begin{pmatrix} M_{F_{11}} & M_{F_{12}} \\ M_{F_{21}} & M_{F_{22}} \end{pmatrix}.$$

Here, F^t denotes the pointwise transpose of F. We define the constant functions $e_1, e_2 \in \mathcal{H}$, where $e_j(z) = e_j$. We compute

$$\begin{aligned}\left\langle F(T)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, F(T)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix}\begin{pmatrix} F_{11} \\ F_{21}\end{pmatrix} \\ \begin{pmatrix} F_{12} \\ F_{22}\end{pmatrix}\end{pmatrix}, \begin{pmatrix}\begin{pmatrix} F_{11} \\ F_{21}\end{pmatrix} \\ \begin{pmatrix} F_{12} \\ F_{22}\end{pmatrix}\end{pmatrix}\right\rangle \\ &= \lambda\left(\begin{pmatrix} F_{11}(z) \\ F_{21}(z)\end{pmatrix}\begin{pmatrix} F_{11}(w)^* & F_{21}(w)^*\end{pmatrix} + \begin{pmatrix} F_{12}(z) \\ F_{22}(z)\end{pmatrix}\begin{pmatrix} F_{12}(w)^* & F_{22}(w)^*\end{pmatrix}\right) \\ &= \lambda\left(F(z)F(w)^*\right) .\end{aligned}$$

Also

$$\left\langle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right\rangle = \lambda\left(e_1 e_1^*\right) + \lambda\left(e_2 e_2^*\right) = \lambda(I) .$$

We combine these equations to get

$$\begin{aligned}\left\langle (I - F(T)^*F(T))\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right\rangle &= \lambda(I - F(z)F(w)^*) \\ &= \lambda\left(\frac{1}{\rho^2} - F(z)F(w)^*\right) - \left(\frac{1}{\rho^2} - 1\right)\lambda(I) < 0 .\end{aligned}$$

Therefore $\|F(T)\| > 1$. However, F is unitary on B, so $\|F\| = 1$. We therefore conclude that π is not completely contractive. □

Later on in this section, we will need to work with matrix valued Herglotz representations, so we will need some results about matrix-valued measures. Given a compact Hausdorff space X, an $m \times m$ matrix-valued measure

$$\mu = \left(\mu_{jl}\right)_{j,l=1}^{m}$$

is an $m \times m$ matrix whose entries μ_{jl} are complex-valued Borel measures on X. The measure μ is positive (we write $\mu \geq 0$) if for each function $f : X \to \mathbb{C}^m$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix},$$

we have

$$0 \leq \int_X f^* d\mu\, f := \sum_{j,l} \int_X \overline{f_j} f_l d\mu_{jl} .$$

The positive measure μ is bounded by $M > 0$ if

$$MI_m - \left(\mu_{jl}(X)\right) \geq 0$$

is positive semi-definite, where I_m is the $m \times m$ identity matrix.

Lemma 3.5.3. *The $m \times m$, matrix-valued measure μ is positive if and only if for each Borel set ω the $m \times m$ matrix*

$$\left(\mu_{jl}(\omega)\right)$$

is positive semi-definite.

Further, if there is a κ so that each diagonal entry $\mu_{jj}(X) \leq \kappa$, then each entry μ_{jl} of μ has total variation at most κ. Particularly, if μ is bounded by M, then each entry has variation at most M.

Proof. First, suppose μ is positive.

If $C(X)$ denotes the continuous complex-valued functions on X, let $f \in C(X)$ be arbitrary, and choose a $c \in \mathbb{C}^m$. Our choice of $c \in \mathbb{C}^m$ induces a linear functional $\Phi_c \in C(X)^*$, given by

$$\Phi_c(f) = \sum_{j,l} \int_X \overline{c_j} c_l f \, d\mu_{jl} .$$

This functional must be positive, so $\sum_{j,l} \overline{c_j} c_l \mu_{jl}$ is a positive measure on X. If ω is a Borel set, then

$$\left\langle \left(\mu_{jl}(\omega)\right) c,\, c \right\rangle = \sum_{j,l} \overline{c_j} c_l \mu_{jl}(\omega) \geq 0 .$$

Since this holds for any $c \in \mathbb{C}^m$, we can see that the matrix $\left(\mu_{jl}(\omega)\right)$ is positive. By setting $c = e_j$ (the j-th standard basis vector), we also see that μ_{jj} is a positive linear functional.

Conversely, suppose that for all Borel sets ω, the matrix $\left(\mu_{jl}(\mu)\right)$ is positive. We know that any continuous function can be approximated by a sequence of step functions,

$$f^{(n)} = \sum_j v_j^{(n)} \chi_{\omega_j^{(n)}} \to f\,,$$

where $v_j^{(n)} \in \mathbb{C}^m$ and $\chi_{\omega_j^{(n)}}$ is the indicator function for a Borel set $\omega_j^{(n)}$, for all j and n. We can now see that

$$0 \leq \sum_j \left\langle \mu(\omega_j^{(n)}) v_j, v_j \right\rangle = \int_X f^{(n)*} d\mu\, f^{(n)} \to \int_X f^* d\mu\, f\,,$$

so the measure is positive.

Finally, suppose that each diagonal entry $\mu_{jj}(X) \leq \kappa$. For $g \in C(X)$, let $\mu_{jl}(g)$ denote $\int_X g\, d\mu_{jl}$, and

$$\lambda = -\frac{\overline{\mu_{jl}(g)}}{\left|\mu_{j,l}(g)\right|} \left\|g\right\|_\infty\,.$$

Now,

$$\begin{aligned} 0 &\leq \int_X \left\langle \begin{pmatrix} d\mu_{jj} & d\mu_{jl} \\ d\mu_{lj} & d\mu_{ll} \end{pmatrix} \begin{pmatrix} \lambda \\ g \end{pmatrix}, \begin{pmatrix} \lambda \\ g \end{pmatrix} \right\rangle \\ &= \left\|g\right\|_\infty^2 \mu_{jj}(X) + \mu_{ll}\left(\left|g\right|^2\right) - 2\left|\mu_{jl}(g)\right| \left\|g\right\|_\infty \\ &\leq 2\left(\left\|g\right\|_\infty^2 \kappa - \left|\mu_{jl}(g)\right| \left\|g\right\|_\infty\right)\,. \end{aligned}$$

We can now see that $\left|\mu_{jl}(g)\right| \leq \left\|g\right\|_\infty \kappa$, so μ_{jl} is a bounded linear functional on $C(X)$ with norm at most κ, so the variation of μ_{jl} is at most κ.

If the measure is bounded by M, then

$$M - \mu_{jj}(X) = \left\langle (MI_m - (\mu(X)))\, e_j, e_j \right\rangle \geq 0\,,$$

so $\mu_{jj}(X) \leq M$, therefore the variation of μ_{jl} is at most M. □

Lemma 3.5.4. *If μ^n is a sequence of positive $m \times m$ matrix-valued measures on X which are all bounded above by M, then μ^n has a weak-∗ convergent subsequence. That is, there exists a positive $m \times m$ matrix-valued measure μ, such that for each pair of continuous functions $f, g : X \to \mathbb{C}^m$,*

$$\sum_{j,l} \int_X f_l \overline{g_j} d\mu_{jl}^{n_k} \to \sum_{j,l} \int_X f_l \overline{g_j} d\mu_{jl}\,.$$

Proof. By the previous lemma, we know that the measures μ^n are bounded in variation by M. Hence, by the Banach-Alaoglu theorem, there is a subsequence (which we also write μ^n) such that each μ_{jl}^n converges in the weak-∗ topology to some μ_{jl} with variation at most M.

Let $f : X \to \mathbb{C}^m$ be continuous. We have

$$0 \le \int_X f^* d\mu^n f = \sum_{j,l} \int_X f_j^* f_l d\mu_{jl}^n \to \sum_{j,l} \int_X f_j^* f_l d\mu_{jl} = \int_X f^* d\mu\, f\,,$$

so μ must be a positive measure. Also, for any vector $c \in \mathbb{C}^m$, thought of as a constant function, we have

$$0 \le M\,\|c\|^2 - \left\langle \left(\mu_{jl}^n(X)\right) c,\, c\right\rangle \to M\,\|c\|^2 - \left\langle \left(\mu_{jl}(X)\right) c,\, c\right\rangle\,.$$

Therefore, μ is bounded above by M. □

Lemma 3.5.5. *If μ is a positive $m \times m$ matrix-valued measure on X, then the diagonal entries, μ_{jj} are positive measures. Further, with $\nu = \sum_j \mu_{jj}$, there exists an $m \times m$ matrix-valued function $\Delta : X \to M_m(\mathbb{C})$ so that $\Delta(x)$ is positive semi-definite for each $x \in X$ and $d\mu = \Delta d\nu$ – that is, for each pair of continuous functions, $f,\, g : X \to \mathbb{C}^m$,*

$$\sum_{j,l} \int_X \overline{g_j} f_l d\mu_{jl} = \sum_{j,l} \int_X \overline{g_j}\, \Delta_{jl} f_l d\nu\,.$$

Proof. We know from Lemma 3.5.3 on page 43 that if ω is a Borel set, and $\mu_{jj}(\omega) = 0$ for all j, then $\mu_{jl}(\omega) = 0$ for all j and l. Therefore, μ_{jl} is absolutely continuous with respect to ν, so the Radon-Nikodým theorem tells us that there is a ν-measurable function Δ_{jl} such that $d\mu_{jl} = \Delta_{jl} d\nu$.

If we fix a vector $c \in \mathbb{C}^m$, then by Lemma 3.5.3, we can see that for each Borel set ω,

$$0 \le \left\langle \left(\mu_{jl}(\omega)\right) c,\, c\right\rangle = \left\langle \left(\left(\int_\omega \Delta_{jl}(x)\, d\nu(x) \right) c,\, c \right) \right\rangle = \int_\omega \sum_{jl} \overline{c_j} c_l \Delta_{jl}(x)\, d\nu(x)\,.$$

Therefore $\langle \Delta(x)\, c,\, c\rangle \ge 0$ almost everywhere with respect to ν.

Now choose a countable dense subset $\left\{c^i\right\} \subseteq \mathbb{C}^m$. For each i we have a set E_i such that $\left\langle \Delta(x)\, c^i,\, c^i\right\rangle \ge 0$ on E_i and $X \backslash E_i$ is ν-null. We can now see that the set

$$X \backslash E := X \backslash \left(\bigcap_{i \in \mathbb{N}} E_i \right) = \bigcup_{i \in \mathbb{N}} X \backslash E_i$$

is a countable union of ν-null sets, so is ν-null. Also, on E, $\left\langle \Delta(x)\, c^i,\, c^i\right\rangle \ge 0$ for all c^i. Since $\left\{c^i\right\}$ is dense in $\mathbb{C}^m$, we know that $\langle \Delta(x)\, c,\, c\rangle \ge 0$ for all $c \in \mathbb{C}^m$ on E, so $\Delta(x)$ is positive for ν-almost all[4] $x \in X$. □

A key result of this section is the existence of a test-function-like realisation for well behaved matrix-valued inner functions. In some ways, this is a partial converse to Theorem 3.5.2 on page 40.

[4] The statement of the theorem called for $\Delta(x)$ to be positive for all $x \in X$. We can easily achieve this by letting $\Delta(x) = 0$ for $x \in X \backslash E$.

Proposition 3.5.6. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R, F is unitary valued on B, and $F(b) = 0$. If $\rho_F = 1$ and if $S \subseteq R$ is a finite set, then there exists a probability measure μ on Π and a positive kernel $\Gamma : S \times S \times \Pi \to \mathbb{C}$ so that*

$$1 - F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right)\Gamma(z, w; p)d\mu(p)\,.$$

Proof. Since $\rho_F = 1$, we can choose a sequence $0 < \rho_n < 1$ such that $\rho \to 1$, and $1-\rho_n^2 F(z)F(w)^* \in C$. Therefore, for each n, there are vector-valued functions $H_{n,j}$, and functions $\eta_{n,j} \in \mathcal{B}H^\infty(X)$ analytic in a neighbourhood of R, such that

$$1 - \rho_n^2 F(z)F(w)^* = \sum_j H_{n,j}(z)\left(1 - \eta_{n,j}(z)\overline{\eta_{n,j}(w)}\right)H_{n,j}(w)^*\,. \tag{3.5.1}$$

Recall Equation (3.2.6) (found in the proof of Theorem 3.2.10 on page 25). We can assume, without loss of generality, that $\eta_{n,j}(b) = 0$. This is because we can post-compose $\eta_{n,j}$ with a Möbius transformation m, so that $m \circ \eta_{n,j}(b) = 0$. Equation (3.2.6) then shows that we can then write

$$1 - \eta_{n,j}(z)\overline{\eta_{n,j}(w)} = h(z)\left(1 - \left(m \circ \eta_{n,j}(z)\right)\overline{\left(m \circ \eta_{n,j}(z)\right)}\right)\overline{h(w)}$$

for some function h.

By Theorem 3.2.10 on page 25, we have a test function realisation for each $\eta_{n,j}$; that is to say, for each n and j, we have a probability measure $\nu_{n,j}$ on Π, and a function $h_{n,j}(z, p)$ (analytic in z and $\nu_{n,j}$-measurable in p) such that

$$1 - \eta_{n,j}(z)\overline{\eta_{n,j}(w)} = \int_\Pi h_{n,j}(z, p)\left(1 - \psi_p(z)\overline{\psi_p(w)}\right)\overline{h_{n,j}(w, p)}d\nu_{n,j}(p)\,. \tag{3.5.2}$$

Since the $\eta_{n,j}$s and ψ_ps disappear at b, we know that

$$1 = \int_\Pi h_{n,j}(b, p)\overline{h_{n,j}(b, p)}d\nu_{n,j}(p)\,.$$

If

$$\nu_n = \sum_j \nu_{n,j}\,,$$

then by the Radon-Nikodým theorem, there is a non-negative function $u_{n,j}(p)$ such that

$$d\nu_{n,j}(p) = u_{n,j}(p)d\nu_n\,. \tag{3.5.3}$$

If we let

$$\Gamma_n(z, w; p) = \sum_j u_{n,j}(p)H_{n,j}(z)h_{n,j}(z, p)\overline{h_{n,j}(w, p)}H_{n,j}(w)^*\,,$$

then we can substitute equations (3.5.2) and (3.5.3) into (3.5.1), to see that

$$I - \rho_n^2 F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right)\Gamma_n(z, w; p)d\nu_n(p)\,. \tag{3.5.4}$$

The kernel Γ that we have constructed is analytic in z and conjugate analytic in w in a neighbourhood of R, and is positive semidefinite.

We know that for any given z,

$$\sup_{p\in\Pi} \left|\psi_p(z)\right| < 1\,.$$

Since S is finite, we also know that

$$c := \sup_{\substack{p\in\Pi \\ z\in S}} \left|\psi_p(z)\right| < 1\,.$$

Therefore,

$$I \geq I - \rho_n^2 F(z)F(w)^* \geq \left(1 - c^2\right)\int_\Pi \Gamma_n(z, z; p)d\nu_n(p)\,.$$

We define a sequence of $|S| \times |S|$ measures with 2×2 entries, by

$$d\mu_n = \left(\Gamma_n(z, w; p)d\nu_n(p)\right)_{z,w\in S}\,.$$

The diagonals of this sequence of measures are bounded by $1/(1-c^2)$. A positive $k\times k$ matrix, whose diagonal entries are at most C, is bounded above by kCI_k. Since μ_n is bounded above, it must have a convergent subsequence (which we will continue to call μ_n), converging to some μ, with $d\mu = \Gamma d\nu$, for some sesquianalytic positive kernel Γ and some positive real measure ν. We can scale Γ and ν so that ν is a probability measure.

We know that for any fixed $z, w \in S$, $1 - \psi_p(z)\overline{\psi_p(w)}$ is a continuous function of p (by Lemma 3.2.8 on page 24), so we can let n tend to infinity in equation (3.5.4), giving

$$I - F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right)\Gamma(z, w; p)d\nu(p)\,.$$

□

The interested reader may note that we have not used the fact that F is unitary on B in this proof (we include it on the statement of the theorem, to emphasise the connection to Theorem 3.5.2).

Another tool that will prove useful is transfer function representations. For our purposes it will suffice to work with relatively simple colligations. We will define a unitary colligation Σ by $\Sigma = (U, K, \mu)$, where μ is a probability measure on Π, K is a Hilbert space, and U is a unitary linear operator, defined by

$$U = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \mathcal{B} \begin{pmatrix} L^2(\mu) \otimes K \\ \oplus \\ \mathbb{C}^2 \end{pmatrix},$$

where $L^2 \otimes K$ can be regarded as K valued L^2.

We define $\Phi : R \to \mathcal{B}\left(L^2(\mu) \otimes K\right)$ by

$$(\Phi(z)\, f)\,(p) = \psi_p(z)\, f(p)\,.$$

From here, we define the transfer function associated to Σ by

$$W_\Sigma(z) = \mathbf{D} + \mathbf{C}\Phi(z)\,(I - \Phi(z)\mathbf{A})^{-1}\,\mathbf{B}\,.$$

We can see that as $\mathbf{A}$ is a contraction and $\Phi(z)$ is a strict contraction, the inverse in W_Σ exists for any $z \in R$.

Readers may note that this definition is similar to the one given in Condition 3 of Theorem 2.2.1 on page 7. The key difference is that we are now dealing with matrix-valued functions.

Proposition 3.5.7. *The transfer function is contraction valued; that is,* $\|W_\Sigma(z)\| \leq 1$ *for all* $z \in R$. *In fact for all* $z, w \in R$

$$I - W_\Sigma(z)W_\Sigma(w)^* = \mathbf{C}\,(I - \Phi(z)\mathbf{A})^{-1}\,[I - \Phi(z)\Phi(w)^*]\,(I - \Phi(w)\mathbf{A})^{*-1}\,\mathbf{C}^*\,.$$

Proof. The proof is by a messy but straightforward calculation. We know that U is unitary, so $UU^* = I$. By considering the entries of UU^* and I, we see that

$$\mathbf{DB}^* = -\mathbf{CA}^*\,, \quad \mathbf{BB}^* = I - \mathbf{AA}^*\,, \quad \mathbf{DD}^* = I - \mathbf{CC}^*\,.$$

We also note that $\Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1} = (I-\mathbf{A}\Phi(z))^{-1}\,\Phi(z)$. We now calculate

$$
\begin{aligned}
&I - W(z)W(w)^* \\
=&I - \Big[\mathbf{D} + \mathbf{C}\Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1}\,\mathbf{B}\Big]\Big[\mathbf{D}^* + \mathbf{B}^*\,(I-\Phi(w)\mathbf{A})^{*-1}\,\Phi(w)^*\mathbf{C}^*\Big] \\
=&I - \mathbf{D}\mathbf{D}^* - \Big[\mathbf{C}\Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1}\,\mathbf{B}\Big]\Big[\mathbf{B}^*\,(I-\Phi(w)\mathbf{A})^{*-1}\,\Phi(w)^*\mathbf{C}^*\Big] \\
&\;- \mathbf{D}\mathbf{B}^*\,(I-\Phi(w)\mathbf{A})^{*-1}\,\Phi(w)^*\mathbf{C}^* - \mathbf{C}\Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1}\,\mathbf{B}\mathbf{D}^* \\
=&\mathbf{C}\begin{bmatrix} I - \Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1}\,(1-\mathbf{A}\mathbf{A}^*)\,(I-\Phi(w)\mathbf{A})^{*-1}\,\Phi(w)^* \\ +\mathbf{A}^*\,(I-\Phi(w)\mathbf{A})^{*-1}\,\Phi(w)^* + \Phi(z)\,(I-\Phi(z)\mathbf{A})^{-1}\,\mathbf{A} \end{bmatrix}\mathbf{C}^* \\
=&\mathbf{C}\begin{bmatrix} I - (I-\mathbf{A}\Phi(z))^{-1}\,\Phi(z)\,(1-\mathbf{A}\mathbf{A}^*)\,\Phi(w)^*\,(I-\mathbf{A}\Phi(w))^{*-1} \\ +\mathbf{A}^*\Phi(w)^*\,(I-\mathbf{A}\Phi(w))^{*-1} + (I-\mathbf{A}\Phi(z))^{-1}\,\Phi(z)\mathbf{A} \end{bmatrix}\mathbf{C}^* \\
=&\mathbf{C}\,(I-\mathbf{A}\Phi(z))^{-1}\begin{bmatrix} (I-\Phi(z)\mathbf{A})(I-\mathbf{A}^*\Phi(w)^*) \\ -\Phi(z)\,(1-\mathbf{A}\mathbf{A}^*)\,\Phi(w)^* \\ +(I-\Phi(z)\mathbf{A})\mathbf{A}^*\Phi(w)^* \\ +\Phi(z)\mathbf{A}(I-\mathbf{A}^*\Phi(w)^*) \end{bmatrix}(I-\mathbf{A}\Phi(w))^{*-1}\,\mathbf{C}^* \\
=&\mathbf{C}\,(I-\Phi(z)\mathbf{A})^{-1}\,[I-\Phi(z)\Phi(w)^*]\,(I-\Phi(w)\mathbf{A})^{*-1}\,\mathbf{C}^*\,.
\end{aligned}
$$

□

Note that if we define $H(w) = (I-\mathbf{A}^*\Phi(w)^*)^{-1}\,\mathbf{C}^*$, for w fixed, $H(w)^*$ is a function on Π, so we write $H_p(w)^*$. We can see that by considering $L^2(\mu)\otimes K$ as a vector-valued L^2 space, Proposition 3.5.7 on the previous page gives

$$I - W(z)W(w)^* = \int_\Pi \left(1-\psi_p(z)\overline{\psi_p(w)}\right) H_p(z)H_p(w)^* d\mu(p)\,.$$

Proposition 3.5.8. *If $S \subseteq R$ is a finite set, $W : S \to M_2(\mathbb{C})$ and there is a positive kernel $\Gamma : S\times S\times\Pi \to M_2(\mathbb{C})$ such that*

$$I - W(z)W(w)^* = \int_\Pi \left(1-\psi_p(z)\overline{\psi_p(w)}\right)\Gamma(z,w;p)\,d\mu(p) \tag{3.5.5}$$

for all $z, w \in S$, then there exists $G : R \to M_2(\mathbb{C})$ such that G is analytic, $\|G(z)\| \leq 1$ and $G(z) = W(z)$ for $z \in S$. Indeed, there exists a finite-dimensional Hilbert space K (dimension at most $2\,|S|$) and a unitary colligation $\Sigma = (U, K, \mu)$ so that

$$G = W_\Sigma\,,$$

and hence there exists $\Delta : R\times R\times\Pi \to M_2(\mathbb{C})$ a positive analytic kernel such that

$$I - G(z)G(w)^* = \int_\Pi \left(1-\psi_p(z)\overline{\psi_p(w)}\right)\Delta(z,w;p)\,d\mu(p)$$

for all $z, w \in R$.

Proof. The proof uses a lurking isometry argument. This, now commonplace, argument first appeared in [**Agl90b**].

For a fixed $p \in \Pi$, consider the block matrix

$$(\Gamma(z, w; p))_{z,w\in S} \ .$$

This is an $|S| \times |S|$ block matrix with 2×2 entries, so its rank cannot exceed $2\,|S|$. By the matrix-valued version of Kolmogorov's Theorem ([**AM02**, Theorem 2.62]), there must be a $2\,|S|$-dimensional Hilbert space K, and a function $H : S \to L^2(\mu) \otimes \mathcal{B}\left(\mathbb{C}^2, K\right)$ such that

$$\Gamma(z, w; p) = H_p(z)H_p(w)^* \quad \mu\text{-almost everywhere.}$$

We define $\mathcal{E}, \mathcal{F} \subseteq (L^2(\mu) \otimes K) \oplus \mathbb{C}^2$ by

$$\mathcal{E} := \operatorname{Span}\left\{\begin{pmatrix} \overline{\psi_p(w)} H_p(w)^* x \\ x \end{pmatrix} : x \in \mathbb{C}^2, w \in S\right\}$$

$$\mathcal{F} := \operatorname{Span}\left\{\begin{pmatrix} H_p(w)^* x \\ W(w)^* x \end{pmatrix} : x \in \mathbb{C}^2, w \in S\right\} .$$

If we rewrite (3.5.5) as

$$I + \int_\Pi H_p(z)\psi_p(z)\overline{\psi_p(w)}H_p(w)^* d\mu(p) = W(z)W(w)^* + \int_\Pi H_p(z)H_p(w)^* d\mu(p) \,,$$

then this tells us that

$$\left\langle \begin{pmatrix} \overline{\psi_p(w)} H_p(w)^* x \\ x \end{pmatrix}, \begin{pmatrix} \overline{\psi_p(z)} H_p(z)^* y \\ y \end{pmatrix} \right\rangle_{\mathcal{E}} = \left\langle \begin{pmatrix} H_p(w)^* x \\ W(w)^* x \end{pmatrix}, \begin{pmatrix} H_p(z)^* y \\ W(z)^* y \end{pmatrix} \right\rangle_{\mathcal{F}}$$

for all $z, w \in S$ and $x, y \in \mathbb{C}^2$. If we define a mapping $V : \mathcal{E} \to \mathcal{F}$ by

$$V\begin{pmatrix} \overline{\psi_p(w)} H_p(w)^* x \\ x \end{pmatrix} = \begin{pmatrix} H_p(w)^* x \\ W(w)^* x \end{pmatrix} ,$$

then V must be an isometry.

Both $\mathcal{E}$ and $\mathcal{F}$ are finite dimensional, so there must be a unitary $U : \widehat{\mathcal{F}} \to \widehat{\mathcal{E}}$, with $\widehat{\mathcal{E}}, \widehat{\mathcal{F}}$ finite dimensional, such that U^* extends V. Since $(L^2(\mu) \otimes K) \oplus \mathbb{C}^2$ is infinite dimensional, we can embed $\widehat{\mathcal{E}}, \widehat{\mathcal{F}}$ in $(L^2(\mu) \otimes K) \oplus \mathbb{C}^2$, and assume that $U \in \mathcal{B}\left((L^2(\mu) \otimes K) \oplus \mathbb{C}^2\right)$. We now have the colligation (U, K, μ), as required by the theorem.

Write

$$U^* = \begin{pmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{pmatrix} \in \mathcal{B}\begin{pmatrix} L^2(\mu) \otimes K \\ \oplus \\ \mathbb{C}^2 \end{pmatrix} .$$

Since $U^*|_{\mathcal{E}} = V$, we have

$$\begin{pmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{pmatrix} \begin{pmatrix} \overline{\psi_p(w)} H_p(w)^* x \\ x \end{pmatrix} = \begin{pmatrix} H_p(w)^* x \\ W(w)^* x \end{pmatrix} .$$

Written as a system of equations, this says

$$\mathbf{A}^* \underbrace{\overline{\psi_p(w)} H_p(w)^* x}_{\Phi(w)^* H_p(w)^* x} + \mathbf{C}^* x = H_p(w)^* x\,,$$

$$\mathbf{B}^* \overline{\psi_p(w)} H_p(w)^* x + \mathbf{D}^* x = W(w)^* x\,.$$

We rearrange the first equation to give

$$H_p(w)^* x = (I - \mathbf{A}^* \Phi(w)^*)^{-1} \, \mathbf{C}^* x\,.$$

We substitute this into the second equation to give

$$\mathbf{B}^* \Phi(w)^* \, (I - \mathbf{A}^* \Phi(w)^*)^{-1} \, \mathbf{C}^* x + \mathbf{D}^* x = W(w)^* x\,.$$

We therefore conclude that

$$\begin{aligned} W(z) &= \mathbf{D} + \mathbf{C} \, (I - \Phi(z)\mathbf{A})^{-1} \, \Phi(z)\mathbf{B} \\ &= \mathbf{D} + \mathbf{C}\Phi(z) \, (I - \mathbf{A}\Phi(z))^{-1} \, \mathbf{B} = W_\Sigma(z) \end{aligned}$$

for all $z \in S$. □

3.5.2. Uniqueness.

Proposition 3.5.9. *Suppose $F : R \to M_2(\mathbb{C})$ is analytic in a neighbourhood of X, unitary on B, and with a standard zero set. Then there exists a set $S \subseteq R$ with $2n + 3$ elements such that, if $Z : R \to M_2(\mathbb{C})$ is contraction-valued, analytic, and $Z(z) = F(z)$ for $z \in S$, then $Z = F$.*

Proof. Let K^b denote the Fay kernel for R defined in Theorem 3.4.7 on page 35. That is, K^b is the reproducing kernel for the Hilbert space

$$\mathbb{H}^2 := H^2(R, \omega_b)$$

of functions analytic in R with $L^2(\omega_b)$ boundary values. Let $\mathbb{H}^2_2$ denote $\mathbb{C}^2$-valued $\mathbb{H}^2$. Since F is unitary valued on B, the mapping V on $\mathbb{H}^2_2$ given by $VG(z) = F(z)G(z)$ is an isometry. Also, as we will show, the kernel of V^* is the span of

$$\mathfrak{B} := \left\{ K^b(\cdot, a_j)\gamma_j : \ j = 1, \ldots, 2n + 2 \right\},$$

where $F(a_j)^* \gamma_j = 0$ and $\gamma_j \neq 0$; that is, (a_j, γ_j) is a zero of F^*.

We note, for future use, that if φ is a scalar-valued analytic function on a neighbourhood of R, with no zeroes on B, and zeroes $w_1, \ldots, w_n \in R$, all of multiplicity one, and $f \in \mathbb{H}^2$ has roots at all these w_is, then $f = \varphi g$ for some $g \in \mathbb{H}^2$.

Given such a φ, suppose $\psi \in \mathbb{H}^2$ and for all $h \in \mathbb{H}^2$ we have $\langle \psi, \varphi h \rangle = 0$. Since the set

$$\mathfrak{K} := \left\{ K^b(\cdot, w_j) : \ 1 \le j \le n \right\}$$

is linearly independent, we know there is some linear combination

$$f = \psi - \sum_{j=1}^{n} c_j K^b(\cdot, w_j)\,,$$

so that $f(w_j) = 0$ for all j, and so $f = \varphi g$ for some g. Since

$$\left\langle K^b(\cdot, w_j), \varphi h \right\rangle = \overline{\varphi(w_j)\, h(w_j)} = 0$$

for each j and h, it follows that $\langle f, \varphi h\rangle = 0$ for all h. In particular, if $h = g$ (the g we found earlier), then

$$\langle \varphi g, \varphi g\rangle = \langle f, \varphi g\rangle = 0\,,$$

so $f \equiv 0$, and so

$$0 = f = \psi - \sum_{j=1}^{n} c_j K^b(\cdot, w_j)\,. \tag{3.5.6}$$

This tells us that ψ is in the span of $\mathfrak{K}$, so $\mathfrak{K}$ is a basis for the orthogonal complement of $\left\{\varphi h : h \in \mathbb{H}^2\right\}$.

We now find the kernel of V^*. Write $a_{2n+1} = a_{2n+2} = b$. Since $F(b) = 0$, there is a function H analytic in a neighbourhood of X so that $F(z) = (z - b)H(z)$. The function $\varphi(z) = (z - b)\det(H(z))$ satisfies the hypothesis of the preceding paragraph.

Let

$$G := \begin{pmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{pmatrix},$$

where $H = \left(h_{jl}\right)$. Then

$$FG = (z - b)HG = (z - b)\det(H)I\,,$$

where I is the 2×2 identity matrix.

Now, suppose $x \in \mathbb{H}^2_2$ and $V^*x = 0$. Let x_1, x_2 be the co-ordinates of x. For each $g \in \mathbb{H}^2_2$,

$$\begin{aligned} 0 &= \langle Gg, V^*x\rangle \\ &= \langle VGg, x\rangle \\ &= \langle (z-b)\det(H)g, x\rangle \\ &= \langle (z-b)\det(H)g_1, x_1\rangle + \langle (z-b)\det(H)g_2, x_2\rangle\,. \end{aligned}$$

It therefore follows from the discussion leading up to (3.5.6) that both x_1 and x_2 are in the span of

$$\left\{K^b(\cdot, a_j) : 1 \le j \le 2n+2\right\},$$

so

$$x \in \operatorname{Span}\left\{K^b(\cdot, a_j)v : 1 \le j \le 2n+2,\ v \in \mathbb{C}^2\right\}.$$

In particular, there exist vectors $v_j \in \mathbb{C}^2$ such that

$$x = \sum_{j=1}^{2n+2} K^b(\cdot, a_j)\, v_j .$$

We can check that $V^* v K^b(\cdot, a) = F(a)^* v K^b(\cdot, a)$, and $F(b)^* = 0$, so

$$0 = V^* x = \sum_{j=1}^{2n} F(a_j)^* v_j K^b(\cdot, a_j),$$

but the $K^b(\cdot, a_j)$s are linearly independent, so $F(a_j)^* v_j = 0$ for all j. Conversely, if $F(a_j)^* v_j = 0$ then $V^* v_j K^b(\cdot, a_j) = 0$, so the kernel of V^* is spanned by $\mathfrak{V}$.

Now, since V is an isometry, $I - VV^*$ is the projection onto the kernel of V^*, which by the above argument has dimension $2n+2$, so $I - VV^*$ has rank $2n+2$. So, for any finite set $A \subseteq R$, the block matrix with 2×2 entries

$$\begin{aligned} M_A &= \left(\left[\left\langle (I - VV^*)\, K^b(\cdot, w)e_j,\ K^b(\cdot, z)e_l \right\rangle\right]_{j,l=1,2}\right)_{z,w \in A} \\ &= \left((I - F(z)F(w)^*)\, K^b(z, w)\right)_{z,w \in A} \end{aligned}$$

has rank at most $2n+2$. In particular, if $A = \{a_1, \ldots, a_{2n+2}\}$, then M_A has rank exactly $2n+2$. Choose a_{2n+3}, a_{2n+4} distinct from $a_1, \ldots, a_{2n+2}$ so that

$$S = \{a_1, \ldots, a_{2n+2}, a_{2n+3}, a_{2n+4}\}$$

has $2n+3$ *distinct* points. Since $A \subseteq S$, M_S has rank at least $2n+2$. However, by the above discussion, its rank cannot exceed $2n+2$, so its rank must be exactly $2n+2$.

The matrix M_S is $(4n+6) \times (4n+6)$, (a $(2n+3) \times (2n+3)$ matrix with 2×2 matrices as its entries), and M_S has rank $2n+2$, so must have nullity (that is, kernel dimension) $2n+4$. Further, the subspace

$$\mathcal{L}_1 := \left\{ \underbrace{\begin{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \alpha_{2n+3} \\ 0 \end{pmatrix} \end{pmatrix}}_{=\alpha \otimes e_1} : \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n+3} \end{pmatrix} \in \mathbb{C}^{2n+3} \right\}$$

is $2n+3$ dimensional, so there exists a non-zero $x_1 = y_1 \otimes e_1$ in $\mathcal{L}_1$ which is in the kernel of M_S. Similarly, $\mathcal{L}_2 := \left\{\alpha \otimes e_2 : \alpha \in \mathbb{C}^{2n+3}\right\}$ contains some x_2 in the kernel of M_S.

Let $x = (x_1 \quad x_2)$, so x is the $(4n+6) \times 2$ matrix

$$x = \begin{pmatrix} \begin{pmatrix} (y_1)_1 & 0 \\ 0 & (y_2)_1 \end{pmatrix} \\ \begin{pmatrix} (y_1)_2 & 0 \\ 0 & (y_2)_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} (y_1)_{2n+3} & 0 \\ 0 & (y_2)_{2n+3} \end{pmatrix} \end{pmatrix}.$$

It will be more convenient to refer to 2×2 blocks in x by their corresponding point in S, rather than their number, so we say

$$x(w) = (x_1(w) \quad x_2(w)) = \begin{pmatrix} y_1(w) & 0 \\ 0 & y_2(w) \end{pmatrix}.$$

In this notation, the identity $M_S x = 0$ becomes

$$\sum_{w \in S} K^b(z, w)\, x(w) = F(z) \sum_{w \in S} K^b(z, w)\, F(w)^*\, x(w)$$

for each z.

Now, suppose $Z : R \to M_2(\mathbb{C})$ is analytic, contraction valued, and $Z(z) = F(z)$ for $z \in S$. The operator W of multiplication by Z on $\mathbb{H}^2_2$ is a contraction and

$$W^* K^b(\cdot, w) v = Z(w)^* v K^b(\cdot, w).$$

Given $\zeta \in R$, $\zeta \notin S$, let $S' = S \cup \{\zeta\}$ and consider the decomposition of

$$N_\zeta = \left((I - Z(z)Z(w)^*)\, K^b(z, w)\right)_{z, w \in S'}$$

into blocks labelled by S and $\{\zeta\}$. Thus N_ζ is a $(2n+4) \times (2n+4)$ matrix with 2×2 block entries. The upper left $(2n+3) \times (2n+3)$ block is simply M_S, as $Z(z) = F(z)$ for $z \in S$.

Let

$$x' = \begin{pmatrix} x \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Since N_ζ is positive semi-definite and $M_S x = 0$, it can be shown that $N_\zeta x' = 0$. An examination of the last two entries of the equation $N_\zeta x' = 0$ gives

$$\sum_{w \in S} K^b(\zeta, w) x(w) = Z(\zeta) \sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w). \tag{3.5.7}$$

The left hand side of (3.5.7) is a rank 2, 2×2 matrix at all but countably many ζ, as it is a diagonal matrix whose diagonal elements are of the form

$$\sum_{w \in S} K^b(\zeta, w) y_i(w);$$

that is, linear combinations of $K^b(\zeta, w)$s. If such a function is zero at an uncountable number of ζs, it is identically zero, which is impossible, as the $K^b(\cdot, w)$s are linearly independent and the $y_i(w)$s are not all zero. If the left hand side of (3.5.7) is invertible, the right hand side must be invertible too. Therefore, we can now see that

$$\sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w)$$

is invertible at all but countably many ζ, so

$$\begin{aligned} Z(\zeta) &= \sum_{w \in S} K^b(\zeta, w) x(w) \left(\sum_{w \in S} Z(w)^* K^b(\zeta, w) x(w) \right)^{-1} \\ &= \sum_{w \in S} K^b(\zeta, w) x(w) \left(\sum_{w \in S} F(w)^* K^b(\zeta, w) x(w) \right)^{-1} \\ &= F(\zeta) \end{aligned}$$

at all but finitely many ζ, so $Z = F$. □

We combine some of the preceding results to get the following.

THEOREM 3.5.10. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R, which is unitary-valued on B, and with a standard zero set. If $\rho_F = 1$, then there exists a unitary colligation $\Sigma = (U, K, \mu)$ such that $F = W_\Sigma$, and so that the dimension of K is at most $4n + 6$. In particular, μ is a probability measure on Π and there is an analytic function $H : R \to L^2(\mu) \otimes M_{4n+6,2}(\mathbb{C})$, denoted by $H_p(z)$, so that*

$$I - F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) H_p(z) H_p(w)^* d\mu(p)$$

for all $z, w \in R$.

PROOF. Using Proposition 3.5.9 on page 51, choose a finite set $S \subseteq R$ such that if $G : R \to M_2(\mathbb{C})$ is analytic and contraction valued, and $G(z) = F(z)$ for $z \in S$, then $G = F$. Using Proposition 3.5.6 on page 46, we have a probability measure μ and a positive kernel $\Gamma : S \times S \times \Pi \to M_2(\mathbb{C})$ such that

$$I - F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) \Gamma(z, w; p)\, d\mu(p)$$

for all $z, w \in S$.

By Proposition 3.5.8 on page 49, there exists a unitary colligation $\Sigma = (U, K, \mu)$ so that K is at most $4n+6$ dimensional, and $W_\Sigma(z) = F(z)$ for $z \in S$. However, our choice of S gives $W_\Sigma = F$ everywhere. We know $\Gamma(z, w; p) = H_p(z)H_p(w)^*$ for some H_p by [**AM02**, Thm. 2.62]. □

Theorem 3.5.11. *Suppose F is a 2×2 matrix-valued function analytic in a neighbourhood of R, which is unitary valued on B, with a standard zero set, and $\rho_F = 1$, and is represented as in Theorem 3.5.10. Let $a_{2n+1} = a_{2n+2} = b$, $\gamma_{2n+1} = e_1$, and $\gamma_{2n+2} = e_2$. Then there exists a set E of μ measure zero, such that for $p \notin E$, for each $v \in \mathbb{C}^{4n+6}$, and for $l = 0, 1, \ldots, n$, the vector function $H_p(\cdot)vK^b(\cdot, z_l(p))$ is in the span of $\left\{K^b(\cdot, a_j)\gamma_j\right\}$, where $z_0(p)(= b)$, $z_1(p), \ldots, z_n(p)$ are the zeroes of ψ_p. Consequently, H_p is analytic on R and extends to a meromorphic function on Y.*

Proof. We showed in the proof of Proposition 3.5.9 on page 51 that given a finite $Q \subseteq R$,

$$M_Q = \left((I - F(z)F(w)^*)\, K^b(z, w)\right)_{z,w \in Q}$$

has rank at most $2n+2$, and that the range of M_Q lies in

$$\mathfrak{M} := \operatorname{span}\left\{\left(K^b(z, a_i)\gamma_i\right)_{z \in Q} : i = 1, \ldots, 2n+2\right\}, \tag{3.5.8}$$

thinking of $\left(K^b(z, a_i)\gamma_i\right)_{z \in Q}$ as a column vector indexed by Q.

We then apply Theorem 3.5.10 on the previous page to give

$$M_Q = \left(\int_\Pi H_p(z)\left(1 - \psi_p(z)\overline{\psi_p(w)}\right) K^b(z, w) H_p(w)^*\, d\mu(p)\right)_{z,w \in Q}.$$

For each p, we define an operator $M_p \in \mathcal{B}(\mathbb{H}^2)$ by

$$\left(M_p f\right)(x) = \psi_p(x)\, f(x).$$

Multiplication by ψ_p is isometric on $\mathbb{H}^2$, so $1 - M_pM_p^* \geq 0$, and so $\left(1 - M_pM_p^*\right) \otimes \mathbb{E} \geq 0$, where $\mathbb{E}$ is the $m \times m$ matrix with all entries equal to 1. From the reproducing property of K^b, we see that $M_p^* K^b(\cdot, z) = \overline{\psi_p(z)} K^b(\cdot, z)$. Thus, if Q is a set of m points in R, and c is the vector $\left(K^b(\cdot, w)\right)_{w \in Q}$, then the matrix

$$P_Q(p) = \left\langle\left[(I - M_pM_p^*) \otimes \mathbb{E}\right] c, c\right\rangle = \left(\left[1 - \psi_p(z)\overline{\psi_p(w)}\right] K^b(z, w)\right)_{z,w \in Q} \geq 0.$$

If we set $\widetilde{Q} = Q \cup \{z_j\}$ for any $j = 0, 1, \ldots, n$, then $P_{\widetilde{Q}}(p) \geq 0$. Further, the upper $m \times m$ block equals $P_Q(p)$ and the right $m \times 1$ column is $\left(K^b(z, z_j(p))\right)_{z \in Q}$. Hence, as a vector,

$$\left(K^b(z, z_j(p))\right)_{z \in Q} \in \operatorname{ran} P_Q(p)^{1/2} = \operatorname{ran} P_Q(p),$$

for $j = 0, 1, \ldots, n$.

Since $P_Q \geq 0$,

$$N_Q(p) := \left(H_p(z)\left(1 - \psi_p(z)\overline{\psi_p(w)}\right) K^b(z, w) H_p(w)^*\right)_{z,w\in Q}$$

is also positive semi-definite for each p. If $M_Q x = 0$, then

$$0 = \int_\Pi \left\langle N_Q(p)x, x\right\rangle \, d\mu(p)\,,$$

so that $\left\langle N_Q(p)x, x\right\rangle = 0$ for almost all p. It follows that $N_Q(p)x = 0$ almost everywhere. Choosing a basis for the kernel of M_Q, there is a set E_Q of μ measure zero so that for $p \notin E_Q$, the kernel of M_Q is a subspace of the kernel of $N_Q(p)$. For such p, the range of $N_Q(p)$ is a subspace of the range of M_Q, so the rank of $N_Q(p)$ is at most $2n + 2$.

Further, if we let $D_Q(p)$ denote the diagonal matrix with $(2 \times (4n + 6)$ block) entries given by

$$D_Q(p)_{z,w} = \begin{cases} H_p(z) & z = w \\ 0 & z \neq w \end{cases}.$$

then $N_Q(p) = D_Q(p)\, P_Q(p)\, D_Q(p)^*$. Since $P_Q(p)$ is positive semi-definite, we conclude that the range of $D_Q(p)\, P_Q(p)$ is in the range of M_Q. Therefore, since $\left(K^b(z, z_j(p))\right)_{z\in Q}$ is in the range of $P_Q(p)$, $\left(H_p(z)\, v\, K^b(z, z_j(p))\right)_{z\in Q}$ is in the range of M_Q for every $v \in \mathbb{C}^{4n+6}$, and $j = 0, 1, \ldots, n$.

Now suppose $Q_m \subseteq R$ is a finite set with

$$Q_m \subseteq Q_{m+1}\,, \quad Q_0 = \{a_1, \ldots, a_{2n}, a_{2n+1}(= b)\}\,,$$

and

$$\mathcal{D} = \bigcup_{m\in\mathbb{N}} Q_m$$

a determining set; that is, an analytic function is uniquely determined by its values on $\mathcal{D}$. Since

$$\left(H_p(z)\, v\, K^b(z, z_j(p))\right)_{z\in Q_m} \in \operatorname{ran} M_{Q_m} \subseteq \mathfrak{M}\,,$$

we see that there are constants $c_i^m(p)$ such that

$$H_p(z)\, v\, K^b(z, z_j(p)) = \sum_{i=1}^{2n+2} c_i^m(p)\, K^b(z, a_i)\, \gamma_i\,, \quad z \in Q_m\,. \tag{3.5.9}$$

By linear independence of the $K^b(\cdot, a_i)$s, the $c_i^m(p)$s are uniquely determined when $n = 0, 1, \ldots$ by this formula. Since $Q_{m+1} \supseteq Q_m$, we see that $c_i^{m+1}(p) = c_i^m(p)$ for all m, so there are unique constants $c_i(p)$ such that

$$H_p(z)\, v\, K^b(z, z_j(p)) = \sum_{i=1}^{2n+2} c_i(p)\, K^b(z, a_i)\, \gamma_i\,, \quad z \in \mathcal{D}\,.$$

Now, by considering this equation when $j = 0$, and using the fact that $K^b(\cdot, b) \equiv 1$, we see that H_p agrees with an analytic function on a determining set. We can therefore assume that H_p is analytic for each $p \notin E$, and that (3.5.9) holds throughout R. Also, since the $K^b(\cdot, a_i)$s extend to meromorphic functions on Y, so must H_p. □

3.5.3. Diagonalisation.

LEMMA 3.5.12. *Suppose F is a matrix-valued function on R whose determinant is not identically zero. If there exists a 2×2 unitary matrix U and scalar valued functions $\phi_1, \phi_2 : R \to \mathbb{C}$ such that $F(z)F(w)^* = UD(z)D(w)^*U^*$, where*

$$D := \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix},$$

then there exists a unitary matrix V such that $F = UDV$.

PROOF. We let $V = D(z)^{-1}U^*F(z)$. This is constant, as the hypothesis implies that $D(z)^{-1}U^*F(z) = D(w)^*U^*F(w)^{*-1}$ whenever $F(z)$ and $F(w)$ are invertible. It is unitary as

$$\begin{aligned} V^*V &= F(z)^*UD(z)^{*-1}D(z)^{-1}UF(z) \\ &= F(z)^* \left(F(z)F(z)^*\right)^{-1} F(z) \\ &= I\,, \end{aligned}$$

and

$$\begin{aligned} VV^* &= D(z)^{-1}UF(z)F(z)^*UD(z)^{*-1} \\ &= D(z)^{-1}U^*UD(z)D(z)^*U^*UD(z)^{*-1} \\ &= I\,. \end{aligned}$$

□

THEOREM 3.5.13. *Suppose F is a 2×2 matrix-valued function which is analytic in a neighbourhood of R, unitary valued on B, and has a standard zero set $\left(a_j, \gamma_j\right)$, $j = 1, \ldots, 2n$. Assume further that the $\left(a_j, \gamma_j\right)$ have the property that if h satisfies*

$$h = \sum_{j=1}^{2n} c_j K^b(\cdot, a_j)\gamma_j + v\,,$$

for some $c_1, \ldots, c_{2n} \in \mathbb{C}$ and $v \in \mathbb{C}^2$, and h does not have a pole at $P_1, \ldots, P_n$, then h is constant.

Under these conditions, if $\rho_F = 1$*, then F is diagonalisable, that is, there exists unitary* 2×2 *matrices U, and V and analytic functions* $\phi_1, \phi_2 : R \to \mathbb{C}$ *such that*

$$F = U \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} V = UDV .$$

Proof. By Theorem 3.5.11 on page 56, we may assume that except on a set E of measure zero, if h is a column of some H_p, then $h(\cdot)K^b(\cdot, z_l(s)) \in \mathfrak{M}$ for $l = 0, 1, \ldots, n$.[5]

By hypothesis, h (and so H_p) is constant. From Remark 3.3.2 on page 27, we can assume at least one of the zeroes of ψ_p (say $z_1(p)$) is not b. Thus, using the proof of the second part of Theorem 3.4.8 on page 36, we can show that if h is not zero, then $z_1(p) = a_{j_1(p)}$ for some $j_1(p)$, and h is a multiple of $\gamma_{j_1(p)}$. Thus, every column of H_p is a multiple of $\gamma_{j_1(p)}$.

Theorem 3.5.10 on page 55 gives us

$$I - F(z)F(w)^* = \int_\Pi \left(1 - \psi_p(z)\overline{\psi_p(w)}\right) H_p H_p^* d\mu(p) ,$$

and substituting $w = b$ gives

$$I = \int_\Pi H_p H_p^* d\mu(p)$$

so

$$F(z)F(w)^* = \int_\Pi \psi_p(z)\overline{\psi_p(w)} H_p H_p^* d\mu(p) . \tag{3.5.10}$$

Since the columns of H_p are all multiples of $\gamma_{j_1(p)}$, $H_p H_p^*$ is rank one, and so can be written as $G(p)G(p)^*$ for a single vector $G(p) \in \mathbb{C}^2$. Consequently,

$$F(z)F(w)^* = \int_\Pi \psi_p(z)\overline{\psi_p(w)} G(p)G(p)^* d\mu(p) . \tag{3.5.11}$$

Since $F(a_j)^*\gamma_j = 0$ for all j, (3.5.11) gives

$$0 = \gamma_j^* F(a_j)F(a_j)^*\gamma_j = \int_\Pi \left|\psi_p(a_j)\right|^2 \left\|G(p)^*\gamma_j\right\|^2 d\mu(p) ,$$

so for each j, $\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$ for almost every p. So, apart from a set $Z_0 \subseteq \Pi$ of measure zero, $\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$ for all p and all j. Thus, by defining $G(p) = 0$ for $p \in Z_0$, we can assume that (3.5.11) holds and

$$\overline{\psi_p(a_j)}G(p)^*\gamma_j = 0$$

for all values of p and j.

Let $\Pi_0 := \{p \in \Pi : G(p) = 0\}$. If $p \notin \Pi_0$, then for each j, either $\psi_p(a_j) = 0$ or $G(p)^*\gamma_j = 0$. Remember that $G(p)$ is a multiple of $\gamma_{j_1(p)}$, and no set of $n+1$ of the γ_j all lie on the same line through the origin. Since F has a standard

[5] Here $z_0(p) = b$, $z_1(p), \ldots, z_n(p)$ are the zeroes of ψ_p, and $\mathfrak{M}$ is as defined in (3.5.8) on page 56.

zero set, it has zeroes at b and $\left(a_j, \gamma_j\right)$ for $j = 1, \ldots, 2n$. Each ψ_p has zeroes at b, and n other points (call these $a_{j_1(p)}, \ldots, a_{j_n(p)}$). At the remaining zeroes of F, we must have that $G(p)^*\gamma_j = 0$, so we must have that $\gamma_{j_{n+1}(p)}, \ldots, \gamma_{j_{2n}(p)}$ all lie on the same line through the origin, (perpendicular to $\gamma_{j_1(p)}$).

We have thus partitioned the zeroes of F into two sets (say $a_j \in \mathfrak{I}_1$ if $\psi_p(a_j) = 0$, and $a_j \in \mathfrak{I}_2$ if $G(p)^*\gamma_j = 0$). Since ψ_p only has n zeroes (excluding b), and at most n of the γ_js can lie on the same line, there is no other way to partition the zeroes. We define $\mathfrak{A}_1 = \operatorname{Span}\left(\gamma_{j_1(p)}\right)$ and $\mathfrak{A}_2 = \operatorname{Span}\left(\gamma_{j_{n+1}(p)}\right)$

We assume (without loss of generality) that $H_p H_p^* \mu$ is supported at more than one point[6], and choose another point, $\hat{p}$. We know that $\psi_{\hat{p}}$ must have different roots to ψ_p, so there must be at least one $a_j \in \mathfrak{I}_2$ such that $\psi_{\hat{p}}(a_j) = 0$, so $G(\hat{p})$ is a multiple of $\gamma_j \in \mathfrak{A}_2$ (using the proof of Theorem 3.4.8 again). By the same reasoning as before, this tells us that n of the γ_js are perpendicular to $G(\hat{p})$, so must lie in $\mathfrak{A}_1$. Since F has $2n$ zeroes, half of them must be in $\mathfrak{A}_1$, and half of them must be in $\mathfrak{A}_2$.

If $q \notin \Pi_0$, then by arguing as above, either $G(q) \in \mathfrak{A}_1$ or $G(q) \in \mathfrak{A}_2$, and the zeroes of ψ_q are in $\mathfrak{I}_2$ or $\mathfrak{I}_1$ respectively. Hence, for each p, one of the following must hold:

(0): $G(p) = 0$;
(1): $G(p) \in \mathfrak{A}_1$ and the zeroes of ψ_q are in $\mathfrak{I}_2 \cup \{b\}$;
(2): $G(p) \in \mathfrak{A}_2$ and the zeroes of ψ_q are in $\mathfrak{I}_1 \cup \{b\}$.

Define

$$\begin{aligned} \Pi_0 &= \{p \in \Pi : \text{ (0) holds}\}, \\ \Pi_1 &= \{p \in \Pi : \text{ (1) holds}\}, \\ \Pi_2 &= \{p \in \Pi : \text{ (2) holds}\}. \end{aligned}$$

If $p, q \in \Pi_1$ then ψ_p and ψ_q are equal, up to multiplication by a unimodular constant, so we choose a $p^1 \in \Pi_1$ and define $\psi_1 = \psi_{p^1}$, so $\psi_p\overline{\psi_p} = \psi_1\overline{\psi_1}$ for all $p \in \Pi_1$. We do the same for Π_2. We substitute this into (3.5.10) to get

$$F(z)F(w)^* = h_1\psi_1(z)\overline{\psi_1(w)}h_1^* + h_2\psi_2(z)\overline{\psi_2(w)}h_2^*,$$

where $h_j \in \mathfrak{A}_j$. We see that h_1, h_2 is an orthonormal basis for $\mathbb{C}^2$, so we can apply Lemma 3.5.12 on page 58, and the result follows. □

3.6. The counterexample

We now have all the tools we need to prove Theorem 3.0.5, as introduced at the beginning of the chapter. First, we constructed $\Psi_{S,\mathbf{p}}$ in Lemma 3.3.7,

[6]This is no loss of generality, as a measure supported at only one point would make F diagonal automatically. We can also assume that $\psi_{\hat{p}}$ is not a scalar multiple of ψ_p, for precisely the same reason.

which is always a 2×2 matrix-valued inner function. We then showed, in Lemma 3.3.8 on page 31, that there was a sequence $\Psi_{S_m,\mathbf{p}}$, such that each term had a standard zero set, with $S_m \neq S_0$ for all m, and such that both $S_m \to S_0$ and $\Psi_{S_m,\mathbf{p}} \to \Psi_{S_0,\mathbf{p}}$ as $m \to \infty$. We showed in Theorem 3.4.8 on page 36, that if the zeroes $\left(a_j, \gamma_j\right)$ of $\Psi_{S_m,\mathbf{p}}$ are close enough to the zeroes of $\Psi_{S_0,\mathbf{p}}$ (they would be, for m large enough, say $m = \mathbf{M}$) then any $\mathbb{C}^2$-valued meromorphic function of the form

$$h(z) = \sum_{j=1}^{2n} c_j K^b(z, a_j)\gamma_j + v$$

with no poles at $P_1, \ldots, P_n$ must be constant. Thus, we take $\Psi = \Psi_{S_{\mathbf{M}},\mathbf{p}}$. Theorem 3.5.13 on page 58 then tells us that if $\rho_\Psi = 1$, then Ψ is diagonalisable. Thus, if Ψ is not diagonalisable, then $\rho_\Psi < 1$. If $\rho_\Psi < 1$, Theorem 3.5.2 on page 40 tells us that there is an operator $T \in \mathcal{B}(H)$ for some H, such that the homomorphism $\pi : \mathcal{R}(X) \to \mathcal{B}(H)$ with $\pi(p/q) = p(T) \cdot q(T)^{-1}$ is contractive but not completely contractive. Therefore, all that remains to be shown is that Ψ is not diagonalisable.

Theorem 3.6.1. *Ψ is not diagonalisable.*

Proof. Suppose, towards an eventual contradiction, that there is a diagonal function D and fixed unitaries U and V such that $D(z) = U\Psi(z)V^*$. D must be unitary valued on B, so must be unitary valued at p_0^-, so by multiplying on the left by $D(p_0^-)^*$, we may assume that $D(p_0^-) = I$. Since $\Psi(p_0^-) = I$, $U = V$.

Let

$$D = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}.$$

Since D is unitary on B, both ϕ_1 and ϕ_2 are unimodular on B. Further, as $\det \Psi$ has $2n+2$ zeroes (up to multiplicity), and a non-constant scalar inner function has at least $n+1$ zeroes, we conclude that either ϕ_1 and ϕ_2 have $n+1$ zeroes each, and take each value in the unit disc $\mathbb{D}$ at least $n+1$ times, or one has $2n+2$ zeroes, and the other is a unimodular constant λ. The latter cannot occur, since

$$0 = \Psi(b) = U^* \begin{pmatrix} \lambda & \cdot \\ \cdot & \cdot \end{pmatrix} U \neq 0\,,$$

which would be a contradiction.

Now, from Lemma 3.3.7 on page 29, $\Psi(\mathbf{p}_1)e_1 = e_1$, so Ue_1 is an eigenvector of $D(\mathbf{p}_1)$, corresponding to the eigenvalue 1, so at least one of the $\phi_j(\mathbf{p}_1)$s is equal to 1. Similarly, Ue_2 is an eigenvector of $D(\varpi(\mathbf{p}_1))$, so at least one of the $\phi_j(\varpi(\mathbf{p}_1))$s is equal to 1. Now, $D(\mathbf{p}_1)$ cannot be a multiple of the identity, as this would mean that one of the ϕ_js was equal to 1 at $\mathbf{p}_1$ and $\varpi(\mathbf{p}_1)$, which is impossible[7]. Therefore, we can assume without loss of generality that

$$D(\mathbf{p}_1) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \qquad D(\varpi(\mathbf{p}_1)) = \begin{pmatrix} \lambda' & 0 \\ 0 & 1 \end{pmatrix},$$

where λ, λ' are unimodular constants. We can see from this that the eigenvectors corresponding to 1 in these matrices are e_1 and e_2, so $Ue_1 = ue_1$, $Ue_2 = u'e_2$ for unimodular constants u, u'. Since D is diagonal, we can assume that $u = u' = 1$, so $U = I$, and $\Psi = D$.

Now, since $S_{\mathbf{M}} \neq S_0$, there exists some i such that $P^{i+} \neq P^{1+}$, so these two projections must have different ranges. However by Lemma 3.3.7,

$$\begin{aligned} P^{i+} &= \Psi(\mathbf{p}_i)\, P^{i+} \\ &= D(\mathbf{p}_i)\, P^{i+} \\ &= \begin{pmatrix} \phi_1(\mathbf{p}_i) & 0 \\ 0 & \phi_2(\mathbf{p}_i) \end{pmatrix} P^{i+}. \end{aligned}$$

This is only possible if $\Psi(\mathbf{p}_i) = I$, but this is impossible, as before. This is our contradiction. Therefore, Ψ is not diagonalisable. □

This concludes the proof of Theorem 3.0.5.

[7] as this would mean it took the value 1 at least once on $B_0, B_2, \ldots, B_n$, and at least twice on B_1, so at least $n + 2$ times.

CHAPTER 4

Constrained Interpolation

This chapter is based on material previously published by the author in [**Pic08b**].

4.1. Introduction

One of the key techniques in Chapter 3 was the test function realisation given in Theorem 3.2.10 on page 25. This is a realisation theorem for holomorphic functions on finitely connected planar domains – it is a realisation of the algebra $H^\infty(R)$. This result was known to Dritschel and McCullough when they wrote [**DM05**], and a special case of it – holomorphic functions on the annulus, $H^\infty(\mathbb{A})$ – is proved in [**DM07**].

Theorem 2.2.2 shows that we can use test functions to solve interpolation problems. When viewed in this light, Theorem 3.2.10 allows us to solve the Nevanlinna-Pick type interpolation problem for planar domains (a simpler solution is given by Abrahamse in [**Abr79**]). Similarly, Agler used a test-function-like realisation to solve the Nevanlinna-Pick interpolation problem on the bidisc (see [**Agl90b**] or [**AM02**]).

In this chapter, we are interested in constrained interpolation problems. These problems have many of the unusual characteristics of harder interpolation problems (they generally require collections of kernels, and have interesting behaviour in the case of matrix valued interpolation, much like interpolation on multiply-connected domains[1]), but are simple enough that we can do calculations explicitly (the kernels are typically rational functions), and we can re-use much of the theory of interpolation on the unit disc.

The constrained interpolation problem, as discussed in [**DPRS07**], is the following: Under what circumstances can we find a bounded, holomorphic function f on the disc, with zero derivative at 0, which takes prescribed values $w_1, \ldots, w_n$ at prescribed points $z_1, \ldots, z_n$? We could, equivalently, define

$$H_1^\infty := \{g \in H^\infty : g'(0) = 0\}$$

[1]McCullough gives a (significantly more complicated) matrix valued generalisation of Abrahamse result ([**Abr79**]) in [**McC96**].

and require that our $f \in H_1^\infty$. The solution, as given in [**DPRS07**], is analogous to the Nevanlinna-Pick theorem; such a function exists if and only if

$$\left((1 - w_i\overline{w_j})k^s(z_j, z_i)\right)_{i,j=1}^n \geq 0 \quad \forall s \in S^2 ,$$

where S^2 denotes the real 2-sphere, and the kernels k^s are a particular class of kernels, parametrised by points s on the sphere[2].

We give a set of test functions for H_1^∞, broadly following the approach of [**AHR08**], via a Herglotz representation for H_1^∞. These test functions turn out to be rational functions, and the set of test functions is parametrised by the sphere. We show, using techniques similar to those in [**DM07**], that our set of test functions is minimal. We also give some indication in Chapter 5 of how these techniques could yield test functions for other types of constrained interpolation problem, although the theory appears to be less elegant in these situations.

We will also introduce the idea of *differentiating kernels*. These are a simple analogue of reproducing kernels, and whilst they are not particularly interesting in and of themselves, they have proved to be a useful tool when working with problems of this sort.

4.2. Differentiating Kernels

It is convenient to introduce *differentiating kernels*. These are along much the same lines as reproducing kernels: We know by Cauchy's integral formula that differentiation is a bounded linear functional on H^2, so for each $x \in \mathbb{D}$, and for each $i = 0, 1, 2, \ldots$, there exists some function $k\left(x^{(i)}, \cdot\right) \in H^2$ such that

$$f^{(i)}(x) = \left\langle f(\cdot), k\left(x^{(i)}, \cdot\right)\right\rangle .$$

The above argument also holds for $H^2(R)$, for any finitely connected planar domain R. In the case of H^2 (that is, $H^2(\mathbb{D})$), we can use Cauchy's integral formula to calculate $k(x^{(i)}, y)$ explicitly. We note that complex contour integration is with respect to $dz = 2\pi i \cdot z\, ds$, where s is normalised arc

[2]They actually give two, different, Nevanlinna-Pick type theorems, but the second is not relevant here.

length measure (the measure on H^2). So, if we let $f \in H^\infty$, then

$$\begin{aligned}
f^{(n)}(x) &= \frac{n!}{2\pi i} \oint_{\mathbb{T}} \frac{f(z)}{(z-x)^{n+1}} dz \\
&= \frac{n!}{\cancel{2\pi i}} \int_{\mathbb{T}} \frac{\cancel{2\pi i}\, z\, f(z)}{(z-x)^{n+1}} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{\overline{z}}{(\overline{z}-\overline{x})^{n+1}} \right)} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{z^{-1}}{(z^{-1}-\overline{x})^{n+1}} \right)} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{z^{n}}{(1-\overline{x}z)^{n+1}} \right)} ds \\
&= \left\langle f(z),\, n! \frac{z^n}{(1-\overline{x}z)^{n+1}} \right\rangle_z .
\end{aligned}$$

Since H^∞ is dense in H^2, this also holds for $f \in H^2$. We can now see that

$$k(x^{(n)},\, y) = n! \frac{y^n}{(1-\overline{x}y)^{n+1}} = \frac{\partial^n}{\partial \overline{x}^n} k(x,\, y)$$

where $k(x,\, y) = (1-\overline{x}y)^{-1}$ is the ordinary Szegő kernel. For brevity, we write $k_{x^{(i)}}$ for the function $k(x^{(i)},\, \cdot)$.

If M_f is the multiplication operator of $f \in H^\infty$ on H^2, these differentiating kernels satisfy

$$M_f^* \frac{k_{x^{(n)}}}{n!} = \sum_{i=0}^{n} \frac{\overline{f^{(i)}(x)}}{i!} \frac{k_{x^{(n-i)}}}{(n-i)!},$$

as

$$\begin{aligned}
\left\langle g,\, M_f^* \frac{k_{x^{(n)}}}{n!} \right\rangle &= \left\langle M_f g,\, \frac{k_{x^{(n)}}}{n!} \right\rangle \\
&= \left\langle fg,\, \frac{k_{x^{(n)}}}{n!} \right\rangle \\
&= \frac{1}{n!} (fg)^{(n)}(x) \\
&= \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(x) g^{(n-i)}(x) \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!} \frac{g^{(n-i)}(x)}{(n-i)!} \\
&= \left\langle g,\, \sum_{i=0}^{n} \frac{\overline{f^{(i)}(x)}}{i!} \frac{k_{x^{(n-i)}}}{(n-i)!} \right\rangle .
\end{aligned}$$

4.3. Test Functions

4.3.1. Zero Mean Probability Measures. We start out by finding a set of test functions for H_1^∞. Since test functions have norm 1 or less, the Möbius transformation

$$m : z \to \frac{1+z}{1-z}$$

(which takes the unit disc to the right half plane), takes test functions to functions with positive real part. Since $H_1^\infty \subseteq H^\infty$, our test functions must have a Herglotz representation (see Theorem 1.1.19 of [**AHR08**]), so if ψ is a test function, $f(z) = m(\psi(z))$, and $f(0) > 0$,[3] then

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w)$$

for some positive measure μ. For our test function to be in H_1^∞, we also need that $\psi'(0) = 0$. We can see that since

$$f'(z) = \psi'(z)\, m'\left(\psi(z)\right)$$

and $m' \neq 0$, we have that $\psi'(0) = 0$ if and only if $f'(0) = 0$.

Now,

$$f'(z) = \int_{\mathbb{T}} \left(\frac{d}{dz}\frac{w+z}{w-z}\right) d\mu(w) = \int_{\mathbb{T}} \frac{2w}{(w-z)^2} d\mu(w)$$

so

$$f'(0) = \int_{\mathbb{T}} \frac{2}{w} d\mu(w) = 2\int_{\mathbb{T}} \overline{w} d\mu(w) = 2\overline{\int_{\mathbb{T}} w d\mu(w)}\,.$$

If μ is a probability measure (which it will later be convenient to assume it is), then the condition that $\psi'(0) = 0$ is equivalent to the condition that $\mathbb{E}(\mu) = 0$, so μ has *zero-mean*. It should be noted here that requiring μ to be a probability measure is equivalent to requiring that $f(0) = 1$, or equivalently still, that $\psi(0) = 0$.

We have proved the following:

Theorem 4.3.1. *The analytic function ψ has $\|\psi\|_\infty \leq 1$, $\psi(0) = 0$, and $\psi'(0) = 0$ if and only if the corresponding measure is a zero-mean probability measure.*

4.3.2. Extreme Directions. We will be using a lot of techniques and definitions from [**AHR08**]. In that paper, they used the convention that if X was a “real function space” in some sense, then X^h is the set of all real functions in X corresponding to holomorphic functions. Here, we will use the convention that X^1 is the set of all real functions in X corresponding to (the real parts of) holomorphic functions with zero derivative at 0, in ways that should be fairly clear.

[3] This condition is not as problematic as it looks.

If $L^2_{\mathbb{R}}(\mathbb{T})$ is defined in the usual way, then $L^{2,1}_{\mathbb{R}}(\mathbb{T})$ is the set of all functions in $L^2_{\mathbb{R}}(\mathbb{T})$ which are the real part of an analytic function in H^2 with $f'(0) = 0$. It is easy to see that $\left(L^{2,1}_{\mathbb{R}}(\mathbb{T})\right)^{\perp} = \operatorname{span}\{\operatorname{Im} z, \operatorname{Re} z\}$, so $L^{2,1}_{\mathbb{R}}(\mathbb{T})$ has (real) co-dimension 2 in $L^2_{\mathbb{R}}(\mathbb{T})$. We let $M_{\mathbb{R}}(\mathbb{T})$ be the space of finite regular real Borel measures[4] on $\mathbb{T}$, and $C_{\mathbb{R}}(\mathbb{T})$ be the space of real continuous functions on $\mathbb{T}$, so that $M_{\mathbb{R}}(\mathbb{T})$ is the dual of $C_{\mathbb{R}}(\mathbb{T})$, under the weak-$*$ and uniform topologies, respectively.

By the convention mentioned above, we define $M^1_{\mathbb{R}}(\mathbb{T})$ and $C^1_{\mathbb{R}}(\mathbb{T})$ as the subspaces of $M_{\mathbb{R}}(\mathbb{T})$ and $C_{\mathbb{R}}(\mathbb{T})$ (respectively), corresponding to holomorphic functions with zero derivative at 0. Then, as in [**AHR08**], we can see that

$$M^1_{\mathbb{R}}(\mathbb{T}) = \{\operatorname{Im} z, \operatorname{Re} z\}^{\perp}$$

and

$$C^1_{\mathbb{R}}(\mathbb{T})^{\perp} = \operatorname{span}\{\operatorname{Im} z\, ds, \operatorname{Re} z\, ds\} ,$$

where $\perp$ denotes the annihilator.

We also need to make use of extreme directions. We say a vector x in a cone C is an *extreme direction* in C if, to have $x = x_1 + x_2$ for some $x_1, x_2 \in C$ we need $x_1 = tx$ and $x_2 = sx$ for some $s, t \geq 0$.

This allows us to formulate the following:

Theorem 4.3.2. *Let* $E = \left\{\mu \in M^1_{\mathbb{R}}(\mathbb{T}) : \mu \geq 0\right\}$. *If* μ *is an extreme direction in* E, *then* μ *is supported at three or fewer points on* $\mathbb{T}$.

Proof. Suppose μ is supported on four or more points in $\mathbb{T}$, and divide the support of $\mathbb{T}$ into four non-empty parts, Δ_1 to Δ_4. Let $\mu_i = \chi_{\Delta_i}\mu$, where χ_Δ is the indicator function on Δ. Let $\mathcal{M} = \operatorname{span}\{\mu_1, \ldots, \mu_4\}$. The dimension of $\mathcal{M}$ is 4, and since $M^1_{\mathbb{R}}(\mathbb{T})$ has co-dimension 2 in $M_{\mathbb{R}}(\mathbb{T})$, we must have that

$$\dim\left(\mathcal{M} \cap M^1_{\mathbb{R}}(\mathbb{T})\right) \geq 2 .$$

Therefore, there must exist a $\nu \in \mathcal{M} \cap M^1_{\mathbb{R}}(\mathbb{T})$ which is linearly independent of μ. Since a measure $\alpha_1\mu_1 + \cdots + \alpha_4\mu_4 \in \mathcal{M}$ is positive whenever $\alpha_1, \ldots, \alpha_4$ are all positive, we can choose an $\epsilon > 0$ small enough that $\mu \pm \epsilon\nu \geq 0$. Therefore, $\frac{1}{2}(\mu \pm \epsilon\nu) \in E$, but

$$\mu = \frac{1}{2}(\mu + \epsilon\nu) + \frac{1}{2}(\mu - \epsilon\nu)$$

so μ is not an extreme direction in E. □

We can combine this with Theorem 4.3.1 on the preceding page, which says that $\mathbb{E}(\mu) = 0$ for $\mu \in E$:

[4]Remember that we can associate a harmonic function to a measure $\mu \in M_{\mathbb{R}}(\mathbb{T})$ via the Poisson kernel.

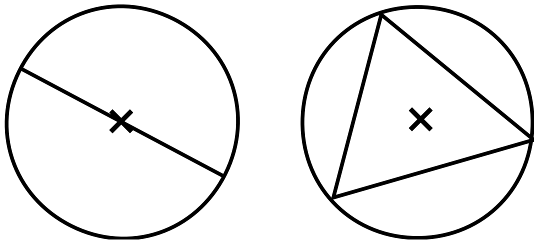

FIGURE 4.3.1. Types of Element in $\widehat{\Theta}$

- If μ is supported at one point of $\mathbb{T}$, then it is clearly impossible to have $\mathbb{E}(\mu) = 0$.
- If μ is supported at two points, then 0 must be in the convex hull of these two points (the line between them), and so the two points must lie opposite each other on the circle, and both have equal weight (if μ is to be a probability measure, this weight must be $\frac{1}{2}$).
- If μ is supported at three points, then these points must be such that 0 is in the *interior* of their convex hull (that is, the interior of the triangle they form; if 0 lies on one of the lines of the triangle, then μ will only be supported on the two points at either end of the line, so this is the degenerate case we had before). We can also see that if μ is a probability measure, then its weights are uniquely determined by its support – the weights are precisely the barycentric co-ordinates of 0, with respect to the three vertices of the triangle.

We now note that, if a measure μ in E is supported on three or fewer points, it is uniquely determined by those points, up to multiplication by a scalar. In particular, if $\mu = t_1\mu_1 + t_2\mu_2$, for some $\mu_1, \mu_2 \in E$, then μ_1 and μ_2 must be supported on a subset of the support of μ, so must be scalar multiples of μ, so we have characterised the extreme directions in E.

Note that we can rescale any non-zero measure in E to a probability measure.

4.3.3. Some Topology. This is a convenient time to talk about the "space" of test functions. We define a set $\widehat{\Theta}$, containing two types of element, as shown in Figure 4.3.1:

- diameters of the circle;
- triangles, with vertices on the circumference of the circle, and the centre of the circle in their interior.

To topologise this set, we say that a sequence of triangles converges to a triangle if its vertices converge, and if we have a sequence of triangles where one of the lines converges to a diameter, then we say the sequence converges

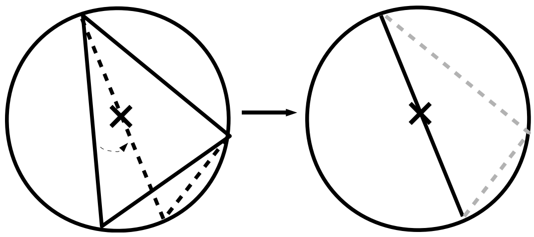

FIGURE 4.3.2. Convergence in $\widehat{\Theta}$

to the diameter, as in Figure 4.3.2. Essentially, the third point on the triangle disappears, which makes sense, as in the case of the zero-mean probability measures above, the weight of the third point would tend to zero. In fact, if we identify points in $\widehat{\Theta}$ with zero-mean probability measures on the circle, as in the discussion following Theorem 4.3.2, we can see that this topology corresponds exactly to the weak-* topology on $M_{\mathbb{R}}(\mathbb{T})$.

As we will see later, if μ is a zero-mean probability measure supported on n points, then μ induces a test function ψ_μ, which will be an n-to-one Blaschke product, with $\psi'_\mu(0) = 0$, $\psi_\mu(0) = 0$, and $\psi_\mu(w) = 1$ precisely when w is in the support of μ. Conversely, if we have such a function ψ, then there is a corresponding zero-mean probability measure μ_ψ, with support $\psi^{-1}(\{1\})$. If we have a test function ψ_μ, corresponding to a $\mu \in \widehat{\Theta}$, then $\widetilde{\psi} := \overline{\psi_\mu(-1)}\psi_\mu$ is a two-or-three-to-one inner function of the type required, and since $\widetilde{\psi}(-1) = 1$, the corresponding measure $\widetilde{\mu}$ is supported at -1. This is useful, as we can safely identify two test functions if one is a constant, unimodular multiple of the other.

If we define an equivalence relation $\sim$ on $\widehat{\Theta}$ by

$$\mu_1 \sim \mu_2 \iff \psi_{\mu_1} = \lambda\psi_{\mu_2} \text{ for some } \|\lambda\| = 1$$

then we can define $\Theta := \widehat{\Theta}/\sim$. By the above reasoning, Θ can be represented as the set of all triangles or diameters in $\widehat{\Theta}$ with a vertex at -1 (i.e, on the leftmost point of the circle). When viewed in this sense, there is only one diameter in Θ.

It is interesting to note that Θ is homeomorphic to S^2. We show (using a certain amount of hand-waving) that Θ is homeomorphic to $\mathbb{C} \cup \{\infty\}$. First, we set the diameter (in Θ) as ∞. This leaves the set of all triangles in Θ. We know that any triangle in Θ can be represented by a triangle with a point at -1 (the left hand side of the circle), so we put point one on the left of the circle, allow point two to vary over the whole top of the circle (corresponding to the real axis), and allow point three to vary over the range of points opposite the arc between point one and point two (corresponding to the imaginary axis) as in Figure 4.3.3 on the next page. As either of these

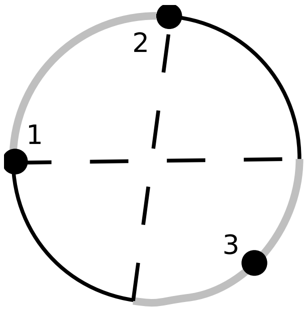

FIGURE 4.3.3. The setup for Θ

points tend towards the edges of their ranges, the triangle tends towards the diameter – which corresponds to ∞.

4.3.4. The Test Functions. If we take the set of probability measures in $M^1_{\mathbb{R}}(\mathbb{T})$, this is a subset of E, and in fact corresponds to E_ρ, in the sense of Lemma 3.4 of [**AHR08**], where $\rho(\mu) = \mu(\mathbb{T})$. We know that E_ρ is convex, and we showed before that its extreme points are given by $\widehat{\Theta}$. E_ρ is compact by the Banach-Alaoglu theorem, so we can apply the Choquet-Bishop-de Leeuw theorem, and we see that for any $\mu \subseteq E_\rho$, there exists a probability measure ν_μ on $\widehat{\Theta}$ such that[5]

$$\mu = \int_{\widehat{\Theta}} \vartheta d\nu_\mu(\vartheta)$$

Now, probability measures in $M^1_{\mathbb{R}}(\mathbb{T})$ correspond to analytic functions f on $\mathbb{D}$ with positive real part, $f'(0) = 0$, and $f(0) = 1$, using the Herglotz representation theorem. We define

$$h_\vartheta(z) := \int_{\mathbb{T}} \frac{w+z}{w-z} d\vartheta(w)$$

and

$$\psi_\vartheta = \frac{h_\vartheta - 1}{h_\vartheta + 1}$$

[5]We are, confusingly but unavoidably, talking about integrating a measure-valued function (the function ϑ), with respect to a measure (the measure ν_μ). Also, the points in the space that ν_μ integrates over (the space $\widehat{\Theta}$) are, themselves, measures (they are measures on $\mathbb{T}$)

for all $\vartheta \in \widehat{\Theta}$. Using the reasoning above, we can write

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w) \\ &= \int_{\mathbb{T}} \frac{w+z}{w-z} \left[\int_{\widehat{\Theta}} (d\vartheta(w))\, d\nu_\mu(\vartheta) \right] \\ &= \int_{\widehat{\Theta}} \left[\int_{\mathbb{T}} \frac{w+z}{w-z} d\vartheta(w) \right] d\nu_\mu(\vartheta) \\ &= \int_{\widehat{\Theta}} h_\vartheta(z) d\nu_\mu(\vartheta)\,. \end{aligned}$$

This gives us a new “Herglotz representation”, which we will call an *Agler-Herglotz representation*:

Theorem 4.3.3. *If f is an analytic function on $\mathbb{D}$ with positive real part, $f'(0) = 0$, and $f(0) > 0$, then there exists some positive real measure ν on $\widehat{\Theta}$ such that*

$$f(z) = \int_{\widehat{\Theta}} h_\vartheta(z) d\nu(\vartheta)\,.$$

We now prove the main result. We will be using Ψ to refer to the set of test functions associated with Θ, so $\Psi := \{\psi_\vartheta : \vartheta \in \Theta\}$):

Theorem 4.3.4. *The two spaces $H^\infty(\mathcal{K}_\Psi)$ and $H_1^\infty(\mathbb{D})$ are isometrically isomorphic, that is, $H^\infty(\mathcal{K}_\Psi) = H_1^\infty(\mathbb{D})$ and $\|\cdot\|_{\mathcal{K}_\Psi} = \|\cdot\|_{H_1^\infty(\mathbb{D})}$*

Proof. One way is simple. Since $\Psi \subseteq H_1^\infty(\mathbb{D})$, we know that $\mathcal{K}_\Psi$ contains the set $\mathcal{K}_1^\infty$ of reproducing kernels given in **[DPRS07]**, so if $\zeta \in H^\infty(\mathcal{K}_\Psi)$, and $\|\zeta\|_{\mathcal{K}_\Psi} \leq 1$, then

$$\left(\left[1 - \zeta(z)\overline{\zeta(w)}\right] k(x,w)\right) \geq 0$$

for all $k \in \mathcal{K}_1^\infty$. Therefore, ζ must be in $H_1^\infty(\mathbb{D})$, with $\|\zeta\|_{H_1^\infty(\mathbb{D})} \leq 1$, so $H^\infty(\mathcal{K}_\Psi) \subseteq H_1^\infty(\mathbb{D})$.

Now, suppose that $\zeta \in H_1^\infty(\mathbb{D})$ and $\|\zeta\|_{H_1^\infty(\mathbb{D})} \leq 1$. For now, we also suppose that $\zeta(0) = 0$. We let

$$f := \frac{1+\zeta}{1-\zeta}\,,$$

so

$$\zeta = \frac{f-1}{f+1}\,.$$

Hence

$$1 - \zeta(z)\overline{\zeta(w)} = 2\frac{f(z) + \overline{f(w)}}{(f(z)+1)(\overline{f(w)}+1)}\,. \tag{4.3.1}$$

We know that f has positive real part, $f(0) = 1$ and $f'(0) = 0$, so we can use our Agler-Herglotz representation, Theorem 4.3.3, and find that there is a measure ν on $\widehat{\Theta}$ such that

$$f = \int_{\widehat{\Theta}} h_\vartheta d\nu(\vartheta)\,.$$

We can then show, using the definition of ψ_ϑ and (4.3.1), that

$$1 - \zeta(z)\overline{\zeta(w)} = \int_{\widehat{\Theta}} \frac{1 - \psi_\vartheta(z)\overline{\psi_\vartheta(w)}}{(f(z)+1)(1-\psi_\vartheta(z))\left(1-\overline{\psi_\vartheta(w)}\right)\left(\overline{f(w)}+1\right)} d\nu(\vartheta)\,.$$

We know that if $\vartheta \in \widehat{\Theta}$, then $\overline{\psi_\vartheta(-1)}\psi_\vartheta \in \Psi$. If we define a positive kernel $\Gamma : \mathbb{D} \times \mathbb{D} \to C_b(\Psi)^*$ by

$$\Gamma(z, w)\alpha = \int_{\widehat{\Theta}} \frac{\alpha\left(\overline{\psi_\vartheta(-1)}\psi_\vartheta\right)}{(f(z)+1)(1-\psi_\vartheta(z))\left(1-\overline{\psi_\vartheta(w)}\right)\left(\overline{f(w)}+1\right)} d\nu(\vartheta)\,,$$

where $\alpha \in C_b(\Psi)$, we can then see that

$$1 - \zeta(z)\overline{\zeta(w)} = \Gamma(z, w)(1 - E(z)E(w)^*)\,,$$

so $\zeta \in H^\infty(\mathcal{K}_\Psi)$ and $\|\zeta\|_{\mathcal{K}_\Theta} \leq 1$.

To see that this holds when $\zeta(0) \neq 0$, simply recall that

$$1 - \left(\frac{\zeta(z)-a}{1-\bar{a}\zeta(z)}\right)\overline{\left(\frac{\zeta(w)-a}{1-\bar{a}\zeta(w)}\right)} = \frac{(1-a\bar{a})\left(1-\zeta(z)\overline{\zeta(w)}\right)}{(1-\bar{a}\zeta(z))\left(1-a\overline{\zeta(w)}\right)}\,.$$

Therefore, $\zeta \in H^\infty(\mathcal{K}_\Psi)$, and $\|\zeta\|_{\mathcal{K}_\Psi} \leq 1$, so $H^\infty(\mathcal{K}_\Psi) = H^\infty_1(\mathbb{D})$ and $\|\cdot\|_{\mathcal{K}_\Psi} = \|\cdot\|_{H^\infty_1(\mathbb{D})}$, as required. □

4.3.5. Minimality. As in **[DM07]**, we show that this set of test functions is minimal, in the sense that there is no closed subset C of Ψ so that C is a set of test functions for H^∞_1. First, we need a lemma.

Lemma 4.3.5. *If, for some measure μ on a space C, and some separable Hilbert space H, $M \in B(H)$, $f \in L^1_{B(H)}(\mu)$, $f(x) \geq 0$ μ-almost everywhere, and $M \geq \int_C f(x)\,d\mu(x)$, then for all scalars $\delta > 0$ there exists some $C' \subseteq C$ and some scalar $N_\delta > 0$, such that $\mu(C - C') < \delta$ and $M \geq N_\delta f(x)$ for all $x \in C'$.*

Proof. Define, for $N > 0$, and $\varphi \in H$,

$$C_N^\varphi = \{x \in C : \langle (M - Nf(x))\varphi, \varphi\rangle \geq 0\}\,,$$

$$C_N = \{x \in C : M \geq Nf(x)\}\,,$$

$$C_0 = \bigcup_{N>0} C_N\,, \qquad C_0^\varphi = \bigcup_{N>0} C_N^\varphi\,.$$

We can see that, for any given $\varphi \in H$, $C - C_0^\varphi$ is a μ-null set. To see this, note that for x to be in $C - C_0^\varphi$, we would need to have $\langle M\varphi, \varphi\rangle = 0$ but

$\langle f(x)\varphi, \varphi\rangle > 0$. However,

$$\langle M\varphi, \varphi\rangle \geq \left\langle \int_C f(x)\, d\mu(x)\, \varphi, \varphi \right\rangle \geq \int_{C - C_0^\varphi} \langle f(x)\, \varphi, \varphi\rangle\, d\mu(x) > 0\,,$$

which is a contradiction.

If Φ is a countable dense subset of the unit ball in H, we can see that

$$C_N = \bigcap_{\varphi\in\Phi} C_N^\varphi$$

and so

$$C - C_0 = \bigcap_{N>0} (C - C_N) = \bigcap_{N>0} \left(\bigcup_{\varphi\in\Phi} \left[C - C_N^\varphi\right] \right)$$
$$= \bigcup_{\varphi\in\Phi} \left(\bigcap_{N>0} \left[C - C_N^\varphi\right] \right) = \bigcup_{\varphi\in\Phi} \left(C - C_0^\varphi\right)\,,$$

which is a countable union of null-sets, and so is a null set.

Consequently, as $N \to 0$, $\mu(C - C_N) \to 0$, and $M \geq N f(x)$ for all $x \in C_N$, so our result is proved. □

Theorem 4.3.6. *No proper closed subset C of Ψ is a set of test functions for H_1^∞.*

Proof. Suppose, towards an eventual contradiction, that C is a proper closed subset of Ψ and $\psi_0 = \psi_{\vartheta_0} \notin C$. Since C is closed, its complement is open, so we can safely assume that ϑ_0 is not a diameter, and not an equilateral triangle[6].

We notice that the differentiating kernels we defined in Section 4.2 on page 64 are rational functions, so we can extend them to the entire Riemann sphere[7], $\mathbb{C} \cup \{\infty\}$. If we do this, then we see that if $x \neq 0$, $k_{x^{(n)}}$ has $n+1$ poles, all at $\overline{x^{-1}}$, and $k_{0^{(n)}}$ has n poles, all at ∞.

The kernels

$$\Delta_\vartheta(z, w) := \left(1 - \psi_\vartheta(z)\overline{\psi_\vartheta(w)}\right) k(w, z)$$

are positive and have rank at most three (k is just the Szegő kernel). To see this, first note that ψ_ϑ has at most three zeroes[8], and that at least two of them must be at zero, as $\psi_\vartheta(0) = 0$ and $\psi'_\vartheta(0) = 0$. Also note that M_ϑ, the operator of multiplication by ψ_ϑ, is an isometry on H^2, so $1 - M_\vartheta M_\vartheta^*$ is the

[6] If we do the calculations, we discover that these two possibilities correspond to the test functions z^2 and $-z^3$, which are inconvenient corner cases

[7] The fact that this is homeomorphic to Θ is not relevant here, and appears to be a coincidence.

[8] If ϑ is a diameter, then it has two zeroes, otherwise it has three.

projection onto

$$\mathfrak{M}_\vartheta := \ker M_\vartheta^* = \operatorname{Span}\left\{k_0, k_{0'}, k_{a_\vartheta}\right\},$$

where a_ϑ is the third zero of ϑ (if ϑ has three zeroes at zero, then $a_\vartheta = 0''$; if ϑ is a diameter, then the span does not include k_{a_ϑ}).

Now,

$$\Delta_\vartheta(z, w) = \left\langle\left(1 - M_\vartheta M_\vartheta^*\right) k_w, k_z\right\rangle := \langle P_\vartheta k_w, k_z\rangle = \langle P_\vartheta k_w, P_\vartheta k_z\rangle\,,$$

and we can view this as a holomorphic function in z, and an antiholomorphic function in w. If we think of the antiholomorphic function as being in the dual of H^2, then

$$\Delta_\vartheta \in H^2 \otimes \left(H^2\right)^* \cong B(H^2)\,.$$

More explicitly, Δ_ϑ defines an operator on H^2 as

$$\Delta_\vartheta f(z) := \int_{\mathbb{T}} \Delta_\vartheta(z, w)\, f(w)\, ds(w)\,,$$

so

$$\begin{aligned}
\langle \Delta_\vartheta f, g\rangle &= \int_{\mathbb{T}}\int_{\mathbb{T}} \overline{g(z)}\, \Delta_\vartheta(z, w)\, f(w)\, ds(w)\, ds(z)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \overline{g(z)}\, \langle P_\vartheta k_w, P_\vartheta k_z\rangle\, f(w)\, ds(w)\, ds(z)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \langle f(w)\, P_\vartheta k_w, g(z)\, P_\vartheta k_z\rangle\, ds(w)\, ds(z)\\
&= \left\langle \int_{\mathbb{T}} f(w)\, P_\vartheta k_w\, ds(w), \int_{\mathbb{T}} g(z)\, P_\vartheta k_z\, ds(z)\right\rangle\\
&= \left\langle \int_{\mathbb{T}} f(w)\, P_\vartheta k_w\, ds(w), \int_{\mathbb{T}} g(z)\, P_\vartheta k_z\, ds(z)\right\rangle_{\mathfrak{M}_\vartheta}\,.
\end{aligned}$$

Hence, we have factorised Δ_ϑ as $A_\vartheta^* A_\vartheta$, where $A_\vartheta : H^2 \to \mathfrak{M}_\vartheta$ is given by

$$A_\vartheta f := \int_{\mathbb{T}} f(w)\, P_\vartheta k_w\, ds(w)\,.$$

We also note now, for use later, that $A_\vartheta^* = I_{\mathfrak{M}_\vartheta}$, the embedding map of $\mathfrak{M}_\vartheta$ into H^2 , as

$$\begin{aligned}
\langle A_\vartheta f, g\rangle &= \left\langle \int_{\mathbb{T}} f(w)\, P_\vartheta k_w \, ds(w), g \right\rangle_{\mathfrak{M}_\vartheta} \\
&= \int_{\mathbb{T}} f(w)\ \langle P_\vartheta k_w, g\rangle_{\mathfrak{M}_\vartheta}\ ds(w) \\
&= \int_{\mathbb{T}} f(w)\ \left\langle P_\vartheta k_w, I_{\mathfrak{M}_\vartheta} g\right\rangle\ ds(w) \\
&= \int_{\mathbb{T}} f(w)\ \left\langle k_w, I_{\mathfrak{M}_\vartheta} g\right\rangle\ ds(w) \\
&= \int_{\mathbb{T}} f(w)\overline{\left(I_{\mathfrak{M}_\vartheta} g\right)(w)} ds \\
&= \left\langle f, I_{\mathfrak{M}_\vartheta} g\right\rangle .
\end{aligned}$$

We choose any set of four points $F = \{z_1, z_2, z_3, z_4\} \in \mathbb{D}$, and consider the classical Nevanlinna-Pick problem, of finding a contractive function $\varphi \in H^\infty$ such that $\varphi(z_i) = \psi_0(z_i)$ for all i. Since $\Delta_0(z, w)$ has rank at most three, the 4×4 matrix

$$\left(\left[1 - \psi_0(z_i)\overline{\psi_0(z_j)}\right] k(z_j, z_i)\right)_{i,j=1}^4$$

must be singular, so the problem has a unique solution, $\varphi = \psi_0$.

Now, if we assume that C is a set of test functions for H_1^∞, then by Theorem 2.2.2 on page 8 there must be a positive kernel $\Gamma : F \times F \to C(C)^*$ such that

$$1 - \psi_0(z_i)\overline{\psi_0(z_j)} = \Gamma(z_i, z_j)\left(1 - E(z_i)E(z_j)^*\right) .$$

Indeed, by Theorem 2.2.1 on page 7, this kernel must extend to the whole of $\mathbb{D} \times \mathbb{D}$. We can rewrite this, in our case, by saying that there exists a measure μ on C, and functions $h_l(z, \cdot) \in L^2(\mu)$, for $l = 1, \ldots, 4$, such that

$$1 - \psi_0(z)\overline{\psi_0(w)} = \int_C \sum_{l=1}^4 h_l(z, \psi)\overline{h_l(w, \psi)}\left(1 - \psi(z)\overline{\psi(w)}\right) d\mu(\psi) . \tag{4.3.2}$$

Multiplying this equation by $k(z, w)$ gives

$$\Delta_0(z, w) = \int_C \sum_{l=1}^4 h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)} d\mu(\psi) .$$

Since Δ_ψ is a positive kernel and a positive operator, when seen as an operator on H^2, as above, we can say that for all l,

$$\Delta_0(z, w) \geq \int_C h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)} d\mu(\psi) .$$

We now know, by Lemma 4.3.5 on page 72, that for any $\delta > 0$, there is a set C', and a constant $c_\delta > 0$ such that $\mu(C - C') < \delta$, and

$$\Delta_0(z, w) \geq c_\delta h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)},$$

for all $\psi \in C'$.

If we use our factorisation of Δ_ϑ from above, we see that

$$A_0^* A_0 \geq c_\delta h_l(z, \psi) A_\psi^* A_\psi \overline{h_l(w, \psi)}$$

and so by Douglas' Lemma, the range of $h_l(\cdot, \psi)A_\psi^*$ is contained in the range of A_0^*. Therefore, there exist constants $c_1, \ldots, c_9$ so that

$$h_l(\cdot, \psi)k_0 = c_1 k_0 + c_2 k_{0'} + c_3 k_{a_0} \tag{4.3.3}$$

$$h_l(\cdot, \psi)k_{0'} = c_4 k_0 + c_5 k_{0'} + c_6 k_{a_0} \tag{4.3.4}$$

$$h_l(\cdot, \psi)k_{a_\psi} = c_7 k_0 + c_8 k_{0'} + c_9 k_{a_0}\,. \tag{4.3.5}$$

By letting δ go to zero, we see that these equations must hold for μ-almost-all $\psi \in C$.

Equation (4.3.3) tells us that $h_l(\cdot, \psi) = c_1 k_0 + c_2 k_{0'} + c_3 k_{a_0}$, as k_0 is constant. We can also see that $h_l(\cdot, \psi)$ must extend meromorphically to the Riemann sphere, as these kernels do so. Equation (4.3.4) tells us that $c_2 = 0$, as otherwise the left hand side of the equation has a triple pole at ∞, but the right hand side has at most only a double pole.

We consider equation (4.3.5) in three cases. Firstly, if ψ has only two zeroes, there is no equation (4.3.5), so $h_l(\cdot, \psi) = c_1 + c_3 k_{a_0}$.

If $a_\psi \neq 0$, then

$$\begin{aligned}
h_l(y, \psi)k_{a_\psi}(y) &= c_7 k_0(y) + c_8 k_{0'}(y) + c_9 k_{a_0}(y)\,, \\
\left[c_1 + c_3 \frac{1}{1 - \overline{a_0} y}\right] \frac{1}{1 - \overline{a_\psi} y} &= c_7 + c_8 y + c_9 \frac{1}{1 - \overline{a_0} y}\,, \\
c_1 + c_3 \frac{1}{1 - \overline{a_0} y} &= c_7 - c_7 \overline{a_\psi} y + c_8 y - c_8 \overline{a_\psi} y^2 + \frac{c_9 - c_9 \overline{a_\psi} y}{1 - \overline{a_0} y}\,, \\
c_1 - c_1 \overline{a_0} y + c_3 &= c_7 - c_7 \overline{a_\psi} y - c_7 \overline{a_0} y + c_7 \overline{a_\psi a_0} y^2 \\
&\quad + c_8 y - c_8 \overline{a_\psi} y^2 - c_8 \overline{a_0} y^2 + c_8 \overline{a_\psi a_0} y^3 \\
&\quad + c_9 - c_9 \overline{a_\psi} y\,.
\end{aligned}$$

Looking at the y^3 coefficient tells us that $c_8 = 0$, looking at the y^2 coefficient then tells us that $c_7 = 0$, and so comparing the constant and y coefficients, we see that

$$\begin{cases} c_1 + c_3 &= c_9 \\ c_1\overline{a_0} &= c_9\overline{a_\psi} \end{cases},$$

$$\begin{cases} c_1 &= c_9\frac{\overline{a_\psi}}{\overline{a_0}} \\ c_3 &= c_9\left(1 - \frac{\overline{a_\psi}}{\overline{a_0}}\right) \end{cases},$$

and so

$$h_l(y,\, \psi) = c_9\left(\frac{\overline{a_\psi}}{\overline{a_0}} + \left(1 - \frac{\overline{a_\psi}}{\overline{a_0}}\right)\frac{1}{1 - \overline{a_0 y}}\right) = c\frac{1 - \overline{a_\psi}y}{1 - \overline{a_0}y}.$$

Alternately, if $a_\psi = 0$, then equation (4.3.5) becomes

$$h_l(y,\, \psi)y^2 = c_7 + c_8 y + c_9\frac{1}{1 - \overline{a_0}y},$$

$$c_1 y^2 + c_3\frac{y^2}{1 - \overline{a_0}y} = c_7 + c_8 y + c_9\frac{1}{1 - \overline{a_0}y},$$

$$c_1 y^2 - c_1\overline{a_0}y^3 + c_3 y^2 = c_7 + c_7\overline{a_0}y + c_8 y + c_8\overline{a_0}y^2 + c_9\,.$$

Looking at the y^3 coefficient tells us that $c_1\overline{a_0} = 0$, so $c_1 = 0$. The y^2 terms then tell us that $c_3 = c_8\overline{a_0}$. We then see that

$$h_l(y,\, \psi) = c_3\frac{1}{1 - \overline{a_0}y} = c\frac{1 - \overline{a_\psi}y}{1 - \overline{a_0}y}$$

as before.

Combining these consequences of equations (4.3.3)–(4.3.5) with (4.3.2) gives a more explicit realisation than the one in (4.3.2), namely,

$$\begin{aligned} 1 - \psi_0(z)\overline{\psi_0(w)} = &\\ \sum_{l=1}^{4}\left(\alpha_l + \frac{\beta_l}{1 - \overline{a_0}z}\right)&\left(\overline{\alpha_l} + \frac{\overline{\beta_l}}{1 - a_0\overline{w}}\right)\left(1 - \psi_\infty(z)\overline{\psi_\infty(w)}\right) \\ &+ \int_{C\setminus\{\infty\}} c(\psi)\left(\frac{1 - \overline{a_\psi}z}{1 - \overline{a_0}z}\right)\left(\frac{1 - a_\psi\overline{w}}{1 - a_0\overline{w}}\right)\left(1 - \psi(z)\overline{\psi(w)}\right)d\mu(\psi) \end{aligned}$$

for some positive $c \in L^1(\mu)$ and some $\alpha_1,\, \beta_1,\, \ldots,\, a_4,\, \beta_4 \in \mathbb{C}$. We know that the ψs are Blaschke products, and we know their roots, so we can write this even more explicitly as

$$1 - z^2\overline{w}^2 \frac{z - a_0}{1 - \overline{a_0}z} \frac{\overline{w} - \overline{a_0}}{1 - a_0\overline{w}} =$$

$$\sum_{l=1}^{4} \left(\alpha_l + \frac{\beta_l}{1 - \overline{a_0}z}\right)\left(\overline{\alpha_l} + \frac{\overline{\beta_l}}{1 - a_0\overline{w}}\right)\left(1 - z^2\overline{w}^2\right)$$

$$+ \int_{C\setminus\{\infty\}} c(\psi) \frac{1 - \overline{a_\psi}z}{1 - \overline{a_0}z} \frac{1 - a_\psi\overline{w}}{1 - a_0\overline{w}} \left(1 - z^2\overline{w}^2 \frac{z - a_\psi}{1 - \overline{a_\psi}z} \frac{\overline{w} - \overline{a_\psi}}{1 - a_\psi\overline{w}}\right) d\mu(\psi) .$$

If we multiply both sides by $(1 - \overline{a_0}z)\,(1 - a_0\overline{w})$ we get

$$(1 - \overline{a_0}z)\,(1 - a_0\overline{w}) - z^2\overline{w}^2\,(z - a_0)\,(\overline{w} - \overline{a_0}) =$$

$$\sum_{l=1}^{4} \left(\alpha_l\,(1 - \overline{a_0}z) + \beta_l\right)\left(\overline{\alpha_l}\,(1 - a_0\overline{w}) + \overline{\beta_l}\right)\left(1 - z^2\overline{w}^2\right)$$

$$+ \int_{C\setminus\{\infty\}} c(\psi)\left(\left(1 - \overline{a_\psi}z\right)\left(1 - a_\psi\overline{w}\right) - z^2\overline{w}^2\left(z - a_\psi\right)\left(\overline{w} - \overline{a_\psi}\right)\right) d\mu(\psi) ,$$

which we can expand to get

$$1 - \overline{a_0}z - a_0\overline{w} + |a_0|^2\, z\overline{w} - z^3\overline{w}^3 + a_0 z^2\overline{w}^3 + \overline{a_0}z^3\overline{w}^2 - |a_0|^2\, z^2\overline{w}^2 =$$

$$\sum_{l=1}^{4} \left\{ \begin{array}{c} \left|\alpha_l + \beta_l\right|^2 - \left|\alpha_l + \beta_l\right|^2 z^2\overline{w}^2 \\ -(\alpha_l + \beta_l)\overline{\alpha_l}a_0\overline{w} + (\alpha_l + \beta_l)\overline{\alpha_l}a_0 z^2\overline{w}^3 \\ -(\overline{\alpha_l} + \overline{\beta_l})\alpha_l\overline{a_0}z + (\overline{\alpha_l} + \overline{\beta_l})\alpha_l\overline{a_0}z^3\overline{w}^2 \\ + |\alpha_l|^2\,|a_0|^2\, z\overline{w} - |\alpha_l|^2\,|a_0|^2\, z^3\overline{w}^3 \end{array} \right.$$

$$+ \int_{C\setminus\{\infty\}} c(\psi) \left[\begin{array}{c} 1 - \overline{a_\psi}z - a_\psi\overline{w} + \left|a_\psi\right|^2 z\overline{w} \\ -z^3\overline{w}^3 + \overline{a_\psi}z^3\overline{w}^2 + a_\psi z^2\overline{w}^3 - \left|a_\psi\right|^2 z^2\overline{w}^2 \end{array} \right] d\mu(\psi) .$$

To get our contradiction, we look at the $\overline{w}$, $z^3\overline{w}^3$ and $z^2\overline{w}^2$ coefficients of this equation. These yield

$$a_0 = \sum_{l=1}^{4} (\alpha_l + \beta_l)\overline{\alpha_l}a_0 + \int_{C\setminus\{\infty\}} c(\psi) a_\psi d\mu(\psi) , \tag{4.3.6}$$

$$1 = \sum_{l=1}^{4} |\alpha_l|^2\,|a_0|^2 + \int_{C\setminus\{\infty\}} c(\psi) d\mu(\psi) , \tag{4.3.7}$$

$$|a_0|^2 = \sum_{l=1}^{4} \left|\alpha_l + \beta_l\right|^2 + \int_{C\setminus\{\infty\}} c(\psi)\left|a_\psi\right|^2 . \tag{4.3.8}$$

We can easily see that (4.3.6) implies

$$|a_0|^2 = \left|\sum_{l=1}^{4} (\alpha_l + \beta_l)\overline{\alpha_l}a_0 + \int_{C\setminus\{\infty\}} c(\psi) a_\psi d\mu(\psi)\right|^2 ,$$

and if we define a Hilbert space $H = \mathbb{C}^4 \oplus L^2(\mu)$, then we can rewrite this as

$$|a_0|^2 = \left| \left\langle \begin{pmatrix} (a_0\overline{\alpha_l})_l \\ c(\psi)^{1/2} \end{pmatrix}, \begin{pmatrix} (\overline{\alpha_l} + \overline{\beta_l})_l \\ c(\psi)^{1/2}\overline{a_\psi} \end{pmatrix} \right\rangle \right|^2 . \tag{4.3.9}$$

Then by the Cauchy-Schwarz inequality,

$$|a_0|^2 \leq \underbrace{\left(\sum_{l=1}^{4} |\alpha_l|^2 |a_0|^2 + \int_{C\setminus\{\infty\}} c(\psi) d\mu(\psi) \right)}_{=1 \text{ by } (4.3.7)} \underbrace{\left(\sum_{l=1}^{4} \left|\alpha_l + \beta_l\right|^2 + \int_{C\setminus\{\infty\}} c(\psi) \left|a_\psi\right|^2 \right)}_{=|a_0|^2 \text{ by } (4.3.8)}$$
$$= |a_0|^2 .$$

Further, the two vectors in (4.3.9) are linearly independent, as $\overline{a_\psi}$ is (the complex conjugate of) the third root of ψ, which is different for each ψ (meaning $c(\psi)^{1/2}$ and $c(\psi)^{1/2}\overline{a_\psi}$ are linearly independent), so the inequality is strict, which is a contradiction[9]. □

The fact that the minimum set of test functions is parametrised by the sphere is interesting, as the set of kernels given in [**DPRS07**] is also parametrised by the sphere, and conjectured to be minimal (that paper contains some partial results, towards this aim).

Similarly, Abrahamse gave (in [**Abr79**]) a set of kernels corresponding to interpolation on a multiply connected (n-holed) domain, parametrised by the n-torus, and conjectured that this set of kernels was minimal (there are some partial results in this direction in [**BC96**]); in [**DM07**] and Note 3.2.11 on page 26, the authors give sets of test functions for n-holed domains, which are also parametrised by the n-torus, and conjectured to be minimal (see Chapter 5).

Perhaps there is some sort of duality between minimal sets of test functions and minimal sets of kernels? A possible counterexample to such a duality may occur with the bidisc; it is well known (see for example [**AM02**]) that only two test functions are needed for the bidisc, whereas in [**MP02**], the authors conjecture that infinitely many kernels are required.

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[9] This is a slight oversimplification. If $c(\psi)$ is non-zero at exactly one point ψ (and that ψ is singular with respect to μ), then this linear independence argument does not hold. However, a simple calculation by equating coefficients (which is omitted) shows that if the vectors are linearly dependent, so $c(\psi)$ is non-zero at exactly one point, then that point must be $\psi_0 \notin C$, which is also a contradiction.

CHAPTER 5

Further Work

5.1. Rational Dilation in Non-Symmetric Domains

Chapter 3 proves that the rational dilation conjecture fails on symmetric, multiply connected domains. However, it says nothing about non-symmetric domains. Many of the results in Chapter 3 should be equally applicable to non-symmetric domains, however some results have proven particularly difficult to generalise.

Proposition 3.1.4 on page 12 is a simple, but useful lemma. It has a simple proof, which uses the symmetry of R in an elegant way. Thus far, we have been unable to prove Proposition 3.1.4 in any other way.

Theorem 3.3.3 and Lemma 3.3.4 on page 28 both depend heavily on the symmetry of R. This is a problem, as Lemma 3.3.4 is used in the proof that our counterexample $\Psi_{S,\mathbf{p}}$ is analytic.

Additionally, the construction of $\Psi_{S,\mathbf{p}}$ in Section 3.3.2 on page 29 uses two related test functions, that are related by the symmetry of R. We have not been able to find an analogous construction, using unrelated test functions, that satisfies Lemma 3.3.7.

Various results (such as Lemma 3.4.6 on page 34) depend on the symmetry of R to prove that the zeroes and poles of functions lie in particular regions. These results are then used to show that the zeroes and poles of various functions do not intersect. Since the zeroes and poles of randomly chosen functions are unlikely to intersect, it is possible that these results are automatically true almost everywhere, or that we can avoid these types of problems by perturbing variables.

5.2. Minimal Test Functions for $H^\infty(R)$

In Section 3.2.4 we gave a set of test functions for $H^\infty(R)$. We believe that this set of test functions (more specifically, the set given in Note 3.2.11 on page 26) is minimal. However, at the time, our main research aim was to find counterexamples to the rational dilation conjecture, so we made no attempt to prove this.

Some of the techniques we have used elsewhere may be used to prove the minimality of our test functions. A proof would likely have a similar structure to the proof that our H_1^∞ test functions are minimal (Theorem 4.3.6 on page 73).

Also, the results of Section 3.5.2 are used to prove that a particular function is uniquely represented, but could potentially be adapted to show that the test functions are uniquely represented, as in the proof of Theorem 4.3.6, and in Proposition 5.3 of [**DM07**], thus proving minimality.

5.3. Matrix-Valued Test Functions

Chapter 3 can, in some ways, be seen as a result about matrix-valued test function realisation; It shows that scalar-valued functions are not sufficient to realise matrix-valued functions on R. The obvious question then is: Can we realise matrix-valued functions using matrix-valued test functions?

This poses various problems. The most serious, is that there is currently no widely accepted definition of a matrix-valued test function. We take most of our definitions from [**DM07**], and we would expect a reasonable definition of matrix-valued test functions to satisfy the major theorems of that paper.

At this point, we are not prepared to suggest a definition, although it is worth noting that many of the results from Section 3.5 are similar to results in [**DM07**].

5.4. Generalised Constrained Test Functions

In Chapter 4 we gave a set of test functions for H_1^∞. However, these techniques could have applications to other types of constrained space. In [**Rag08b**], the author looks at spaces of the form[1] $\mathbb{C} + BH^\infty$. These spaces are a natural generalisation of the space H_1^∞, and the author provides a generalisation of the Nevanlinna-Pick theorem from [**DPRS07**][2].

In Section 4.3.1, we used the Herglotz representation trick to turn a linear equation ($f'(0) = 0$) into a constraint on probability measures (that they have zero mean). Then, in Section 4.3.2 we found the extreme points of the set of constrained probability measures, and used those extreme probability measures to generate our test functions.

Suppose we want to apply these techniques to $\mathbb{C} + BH^\infty$. It is fairly clear that we can come up with a set of linear equations that all functions in $\mathbb{C} + BH^\infty$ must satisfy (functions must be constant at zeroes of B, and have a prescribed number of zero derivatives at repeated zeroes of B). It should

[1] Here, B is a Blaschke product.

[2] In [**DPRS07**], two versions of the Nevanlinna-Pick theorem are given. In [**Rag08b**], the author only generalises the first form; a generalisation of the second form is given in [**BBT08**].

also be fairly easy to turn this set of linear equations into a set of constraints on probability measures.

The difficulty comes when we calculate extreme points. Theorem 4.3.2 should work well enough, so if we have n independent equations, we can say that extreme measures are supported on at most $2n+1$ points. However, the reasoning that followed Theorem 4.3.2 (which showed precisely which such measures would give functions in H_1^∞) relied on a geometric interpretation of the probability constraint, which does not obviously generalise to other types of constraint.

If we can calculate the extreme measures, these should generate test functions in precisely the same way as above. However, the proof that the test functions we have found are minimal, is heavily dependent – perhaps overly dependent – on explicit calculations using the test functions.

5.5. Minimal vs. Extremal

We gave a minimal set of test functions for H_1^∞, and showed that this set was minimal by a long-winded calculation. However, this is not particularly intuitive, and does not obviously generalise. It would seem more natural to use the fact that the Agler-Herglotz representation from Theorem 4.3.3 was parametrised by extreme measures. To do this, we would need some sort of duality between Agler-Herlglotz representations and test function realisations.

We already have some idea what such a duality might look like. The test function realisation for H_1^∞ (Theorem 4.3.4 on page 71) was based on an Agler-Herglotz representation (Theorem 4.3.3) . Similarly, in Theorem 3.2.10 we gave a test function realisation[3] for $H^\infty(R)$. This was based on an Agler-Herglotz representation (found in equation 3.2.5 on page 25.[4]).

In both of these cases, the Agler-Herglotz representation induces a test function realisation (which we call Θ_i). The induced realisations are not quite minimal, as some test functions are scalar multiples of others. However, if we mod out by the equivalence relation $\psi_1 \sim \psi_2 : \psi_1 = \lambda\psi_2$, we get a minimal set of test functions (which we call Θ_m).

Interestingly, in both cases, the equivalence classes of $\sim$ are all circles.

The simplest case is $H^\infty(\mathbb{D})$, where the relevance of circles is clear. The minimal set of test functions is $\Theta_m = \{z\}$, the induced set of test functions is

[3]Here, we are assuming that this test function realisation is minimal, even though this remains to be proved

[4]This is the same as Herglotz Representation 1 in [**AHR08**]

$\Theta_i = \{\lambda z : |\lambda| = 1\}$, and the Herglotz representation can be written as

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w) \\ &= \int_{\mathbb{T}} \frac{1+\lambda z}{1-\lambda z} d\mu(\overline{\lambda}) \\ &= \int_{\Theta_i} \frac{1+\theta(z)}{1-\theta(z)} d\nu(\theta) \,. \end{aligned}$$

In the case of $H^\infty(R)$ and H^∞_1, the relationship between the Agler-Herglotz representation and the test function realisation is similar.

However, it is not clear that there will always be such a link. Our attempts to construct Agler-Herglotz representations from test function realisations have so far been unsuccessful. It also seems unlikely that naïve approaches will work, as $H^\infty(\mathbb{D}^2)$ has a relatively simple test function structure, but has no simple Agler-Herglotz representation[5].

[5] See [**Agl90b**] for a somewhat complicated, abstract Herglotz-like representation

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